# On a certain condition for the projectivization of a leg bundle to become a GKM graph

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## 1. Introduction

This article is the research announcement of the progress work on the leg bundles.

A GKM graph with legs has first appeared in [**KU**] to define the GKM theoretical counterpart of the toric hyperKähler manifold. This involves the $T^n \times S^1$-action on $T^*\mathbb{C}P^n$. Remark that the standard $T^n$-action on $T^*\mathbb{C}P^n$ does not satisfy the GKM condition. However, we define its GKM theoretical counterpart in [**KU**] and apply it to prove the graph equivariant cohomology of some classes of GKM graphs with legs. Motivated by this, in [**KS**], we introduce the leg bundles which are the combinatorial counterparts of the equivariant vector bundles over GKM manifolds. In general, the equivariant vector bundle over a GKM manifold does not satisfy the GKM condition; however, we can define the GKM graph like object for this and define the notion of the projectivization of a complex vector bundle as the purely combinatorial way. In general, a leg bundle may not be the GKM graph but its projectivization may be the GKM graph. So the following problem is the natural problem:

Problem 1.1. *Find the necessary and sufficient conditions when the projectivization of a leg bundle is a GKM graph.*

The purpose of this note is to give a partial answer to this question.

## 2. Leg bundle over a GKM graph and its projectivization

The aim of this section is the quick introduction of a leg bundle over the GKM graph, and the projectivization of a leg bundle (see [**KS**] for details). Throughout of this paper we will use the symbol $|X|$ as the cardinality of the finite set $X$, and the symbol $[r]$ as the set of $\{1, \dots, r\}$ for $r \in \mathbb{N}$. In this paper, we often use the following identification:

$$\mathbb{Z}^n \simeq (\mathfrak{t}^n_{\mathbb{Z}})^* \simeq \mathrm{Hom}(T^n, S^1) \simeq H^2(BT^n) \subset H^*(BT^n) \simeq \mathbb{Z}[x_1, \dots, x_n],$$

where $(\mathfrak{t}^n_{\mathbb{Z}})^*$ is the dual of the weight lattice of the Lie algebra of $T^n$ and $\deg x_i = 2$.

**2.1. Leg bundle over an abstract graph.** Let $\mathcal{V}$ be a set of vertices, and $\mathcal{E}$ be a set of (oriented and possibly multiple) edges in $G$. We denote $G = (\mathcal{V}, \mathcal{E})$. Throughout this paper, we assume that every graph $G$ is connected and finite. We use the following notations:

- $i(e) \in \mathcal{V}$ (resp. $t(e) \in \mathcal{V}$) is the initial (resp. terminal) vertex for $e \in \mathcal{E}$;
- $\overline{e} \in \mathcal{E}$ is the opposite directed graph of $e \in \mathcal{E}$;
- $\mathrm{star}_G(p) := \{e \in \mathcal{E} \mid i(e) = p\}$ is the set of out-going edges from $p \in \mathcal{V}$.

The graph $G = (\mathcal{V}, \mathcal{E})$ is called a *(regular) $m$-valent graph* if $|\mathrm{star}_G(p)| = m$ for every $p \in \mathcal{V}$.

Definition 2.1. Let $G = (\mathcal{V}, \mathcal{E})$ be a graph. The following pair of sets is called a *rank $r$ leg bundle* over $G$:

$$[r]_G := (\mathcal{V}, \mathcal{E} \sqcup \mathcal{V} \times [r]).$$

An element $(p, j) \in \mathcal{V} \times [r]$ is called a *leg* of $[r]_G$ over $p \in \mathcal{V}$. The set of legs over $p$, i.e., $[r]_p := \{(p,1), \ldots, (p,r)\}$ is called the *fiber* of $[r]_G$ over $p$.

The rank $r$ leg bundle $[r]_G$ over $G$ may be regarded as the non-compact graph consisting of the graph $G$ with adding the $r$ non-compact edges, called *legs*, over each vertex of $G$, see Figure 1.

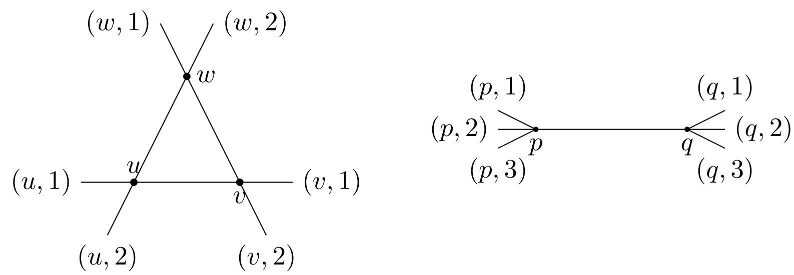


FIGURE 1. The rank 2 leg bundle over the triangle (left) and the rank 3 leg bundle over the edge (right).

**2.2. Leg bundle over a GKM graph.** Let $G = (\mathcal{V}, \mathcal{E})$ be an $m$-valent graph. For $n \leq m$, a function $\alpha : \mathcal{E} \to (\mathfrak{t}^n_{\mathbb{Z}})^*$ satifying the following conditions (1)–(3) is called an *axial function*:

(1) $\alpha(e) = \pm\alpha(\overline{e})$ for every edge $e \in \mathcal{E}$;
(2) every two distinct elements in $\alpha(\mathrm{star}_G(p)) = \{\alpha(e) \in (\mathfrak{t}^n_{\mathbb{Z}})^* \mid e \in \mathrm{star}_G(p)\}$ are linearly independent, i.e., *pairwise linearly independent* (or 2-*independent* for short), for every $p \in \mathcal{V}$;
(3) there is a bijection $\nabla_e : \mathrm{star}_G(i(e)) \to \mathrm{star}_G(t(e))$ for every $e \in \mathcal{E}$ such that
   (a) $\nabla_{\overline{e}} = \nabla_e^{-1}$;
   (b) $\nabla_e(e) = \overline{e}$;
   (c) $\alpha(\nabla_e(e')) - \alpha(e') \equiv 0 \mod \alpha(e)$ for every $e, e' \in \mathrm{star}_G(p)$.

The condition (3)-(c) is called a *congruence relation* on $e \in \mathcal{E}$. The collection $\nabla = \{\nabla_e \mid e \in \mathcal{E}\}$ is called a *connection* on $(G, \alpha)$, and the bijection $\nabla_e$ is also called a *connection* on the edge $e \in \mathcal{E}$. The above triple $(G, \alpha, \nabla)$ is called a *GKM graph*, or an $(m, n)$*-type GKM graph* if we emphasize the valency of $G$ and the dimension of the target space of $\alpha$ (see e.g. [**GZ01, MMP07, DKS22**]).

DEFINITION 2.2 (Leg bundle over a GKM graph). Let $\Gamma = (G, \alpha, \nabla)$ be an $(m, n)$-type GKM graph. We call $\xi$ a *(rank r) leg bundle* over $\Gamma$ if the following data is given for $[r]_G$:

(1) we assign the element $\xi^j_p \in (\mathfrak{t}^n_{\mathbb{Z}})^*$ to every leg $(p, j)$, called a *weight* on $(p, j)$;
(2) there is the permutation $\sigma^\xi_e : [r]_{i(e)} \to [r]_{t(e)}$ for every edge $e \in \mathcal{E}$ that satisfies the following congruence relation:

$$\xi^{\sigma_e(j)}_{t(e)} - \xi^j_{i(e)} \equiv 0 \mod \alpha(e).$$

We also call the collection $\sigma^\xi := \{\sigma^\xi_e \mid e \in \mathcal{E}\}$ a *connection* on $\xi$. A rank 1 leg bundle over $\Gamma$ is called a *line bundle* over $\Gamma$. For a line bundle $\xi$ over $\Gamma$, the connection $\sigma_\xi$ is uniquely determined.

By forgetting legs and their weights, we can define the projection $\pi : \xi \to \Gamma$, see Figure 2.

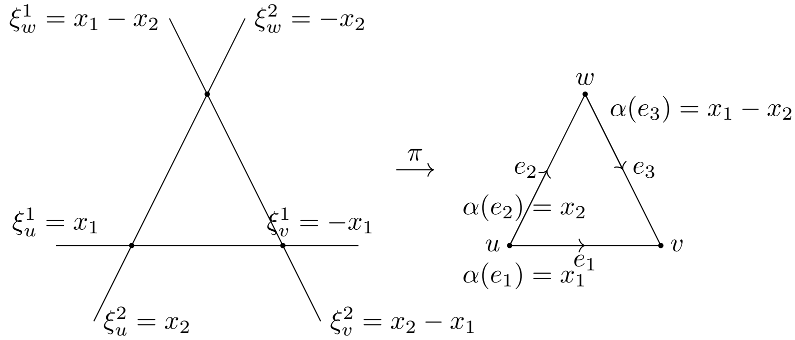


FIGURE 2. The right graph $\Gamma = (G, \alpha, \nabla)$ is the GKM graph satisfying $\alpha(\overline{e}) = -\alpha(e)$. The left labeled graph $\xi$ is the rank 2 leg bundle over $\Gamma$. Note that the connection $\sigma_\xi$ is uniquely determined.

**2.3. Projectivization of a leg bundle.** Let $\Gamma = (G, \alpha, \nabla)$ be an $(m, n)$-type GKM graph and $\xi$ be its rank $(r+1)$ leg bundle, where $G = (\mathcal{V}, \mathcal{E})$. We next introduce the projectivization $\Pi(\xi) = (P(\xi), \alpha^{P(\xi)}, \nabla^{P(\xi)})$ of $\xi$.

The underlying graph $P(\xi) := (\mathcal{V}^{P(\xi)}, \mathcal{E}^{P(\xi)})$ is defined as follows:

- The set of vertices is defined by $\mathcal{V}^{P(\xi)} := [r+1]_G$;
- The set of edges $\mathcal{E}^{P(\xi)}$ consists of the following two types of edges:
  - **vertical:** a *vertical edge* $(p, jk)$ connecting two vertices $(p, j), \ (p, k) \in [r+1]_p$ if $j \neq k$, where $p$ runs over $\mathcal{V}$ and $j, k$ run over $[r+1]_p$ with $j \neq k$;
  - **horizontal:** a *horizontal edge* $(e, l)$ for $e \in \mathcal{E}$ and $l \in [r+1]_{i(e)}$ connecting $(i(e), l)$ and $(t(e), \sigma_e^\xi(l))$.

Note that the reversed orientation edge of the vertical edge $(p, jk)$ is $\overline{(p, jk)} = (p, kj)$ and that of the horizontal edge $(e, l)$ is $\overline{(e, l)} = (\overline{e}, \sigma_e^\xi(l))$.

The *label* $\alpha^{P(\xi)} : \mathcal{E}^{P(\xi)} \to (\mathfrak{t}_{\mathbb{Z}}^n)^*$ of the projectivization $\Pi(\xi)$ is defined as follows:

- $\alpha^{P(\xi)}(p, jk) := \xi_p^j - \xi_p^k$, for any vertical edge $(p, jk) \in \mathcal{E}^{P(\xi)}$;
- $\alpha^{P(\xi)}(e, l) := \alpha(e)$, for any horizontal edge $(e, l) \in \mathcal{E}^{P(\xi)}$.

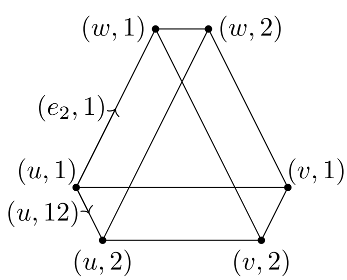


FIGURE 3. The projectivization $P(\xi)$ of the leg bundle $\xi$ in Figure 2. Here, $(u, 12)$ is the vertical edge connecting $(u, 1)$ and $(u, 2)$ and $(e_2, 1)$ is the horizontal edge connecting $(u, 1)$ and $(w, 1)$. For these edges, the labels are defined by $\alpha^{P(\xi)}(u, 12) = \xi_u^1 - \xi_u^2 = x_1 - x_2$ and $\alpha^{P(\xi)}(e_2, 1) = \alpha(e_2) = x_2$.

The *canonical connection* $\nabla^{P(\xi)}$ is defined by the set of the bijective maps

$$\nabla_\epsilon^{P(\xi)} : \mathrm{star}_{P(\xi)}(i(\epsilon)) \longrightarrow \mathrm{star}_{P(\xi)}(t(\epsilon)).$$

such that

- $\nabla_{(u,jk)}^{P(\xi)}(u, jl) = (u, kl)$ for every distinct elements $j, k, l \in [r+1]$;

- $\nabla^{P(\xi)}_{(u,jk)}(e,j) = (e,k)$, where $i(e) = u \in \mathcal{V}$;
- $\nabla^{P(\xi)}_{(e,l)}(u,lk) = (v, \sigma_e(l)\sigma_e(k))$, where $i(e) = u, t(e) = v \in \mathcal{V}$ for every distinct elements $l, k \in [r+1]$;
- $\nabla^{P(\xi)}_{(e,l)}(e',l) = (\nabla_e(e'), \sigma_e(l))$, where $i(e) = i(e') \in \mathcal{V}$,

where we omit $\nabla^{P(\xi)}_{\epsilon}(\epsilon) = \overline{\epsilon}$. The following theorem is straightforward by definition (see [**KS**, Theorem 3.2]).

THEOREM 2.3. *The canonical collection* $\nabla^{P(\xi)} := \{\nabla^{P(\xi)}_{\epsilon} \mid \epsilon \in \mathcal{E}^{P(\xi)}\}$ *satisfies the congruence relations, i.e., it satisfies the conditions to be the connection on* $(P(\xi), \alpha^{P(\xi)})$.

## 3. Whitney sum and Tensor product

Let $\Gamma = (G, \alpha, \nabla)$ be an $(m,n)$-type GKM graph, where $G = (\mathcal{V}, \mathcal{E})$. Let $\xi$ be a rank $r$ and $\eta$ be a rank $r'$ leg bundles over $\Gamma$. In this section, we define the *Whitney sum* $\xi \oplus \eta$ and the *tensor product* $\xi \otimes \eta$.

In order to correspond to the geometrical objects, in this section, we also use the following symbols:

- the symbols $\tau_{\mathbb{C}P^2}$ and $\tau^*_{\mathbb{C}P^2}$ represent the tangent bundle and the cotangent bundle over $\mathbb{C}P^2$ with the standard lifting of the $T^2$-action on $\mathbb{C}P^2$, respectively;
- the symbol $\epsilon_{(k,l)}$ represents the trivial line bundle over $\mathbb{C}P^2$ whose $T^2$-action on the fiber is defined by $(t_1, t_2) \mapsto t_1^k t_2^l$;
- the symbol $\gamma^{\otimes k}$ is the $k$-times tensor product of the tautological line bundle over $\mathbb{C}P^2$ with the standard lifting of the $T^2$-action on $\mathbb{C}P^2$.

All of such equivariant bundles can be described by using the leg bundles. For example, the leg bundle induced from the standard $T^2$-action on the tangent bundle $\tau_{\mathbb{C}P^2}$ is as follows:

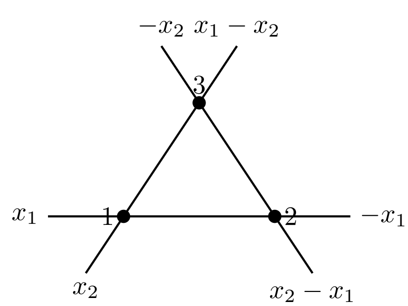


FIGURE 4. Rank 2 leg bundle corresponding to $\tau_{\mathbb{C}P^2}$, where $x_1, x_2 \in (\mathfrak{t}_{\mathbb{Z}})^*$ are the standard basis defined by the coordinate projections. Note that we omit the axial functions on edges.

### 3.1. Whitney sum.

DEFINITION 3.1 (Whitney sum of leg bundles). The *Whitney sum* $\xi \oplus \eta$ is the following rank $r + r'$ leg bundle over $\Gamma$:

(1) the underlying non-compact graph of $\xi \oplus \eta$ is $[r + r']_G$;
(2) the set of legs over the vertex $u \in \mathcal{V}$ is denoted by $[r+r']_u := [r]_u \sqcup [r']_u = \{(u,j) \mid j \in [r]_u\} \sqcup \{(u,j') \mid j \in [r']_u\}$;
(3) for every leg $(u,j) \in [r]_u$ (resp. $(u,j') \in [r']_u$), the label $\xi_u^j$ (resp. $\eta_u^{j'}$) in $(\mathfrak{t}^n_{\mathbb{Z}})^*$ is assigned;
(4) for every edge $e \in \mathcal{E}$, the connection $\sigma_e^{\xi\oplus\eta}$ is defined by $\sigma_e^{\xi\oplus\eta}(i(e), j) := (t(e), \sigma_e^{\xi}(j))$ for $(i(e), j) \in [r]_{i(e)}$ and $\sigma_e^{\xi\oplus\eta}(i(e), j') := (t(e), \sigma_e^{\eta}(j'))$ for $(i(e), j) \in [r']_{i(e)}$.

EXAMPLE 3.2. See Figure 6 from right to left.

### 3.2. Tensor product.

Definition 3.3 (Tensor product of leg bundles). We define the *tensor product* $\xi\eta(=\xi\otimes\eta)$ as follows:

(1) the underlying non-compact graph of $\xi\eta$ is $[rr']_G$;
(2) the set of legs over the vertex $u\in\mathcal{V}$ is denoted by $[rr']_u := \{(u,j,k) \mid j\in[r]_u, k\in[r']_u\} \simeq [r]_u\times[r']_u$;
(3) for every leg $(u,j,k)$, the label $\xi_u^j+\eta_u^k\in(\mathfrak{t}_{\mathbb{Z}}^n)^*$ is assigned;
(4) for every edge $e\in\mathcal{E}$, the connection $\sigma_e^{\xi\eta}$ is defined by $\sigma_e^{\xi\eta}(i(e),j,k) := (t(e),\sigma_e^{\xi}(j),\sigma_e^{\eta}(k))$, where $\sigma_e^{\xi}$ and $\sigma_e^{\eta}$ are connetions on the edge $e$ of $\xi$ and $\eta$ respectively.

Note that if we regard $[rr']_u=[r]_u\times[r']_u$, (4) is nothing but $\sigma_e^{\xi\eta}=\sigma_e^{\xi}\times\sigma_e^{\eta}$. Note that for two line bundles $\zeta$ and $\zeta'$ over $\Pi$, the label of the tensor product $\zeta\zeta'$ can be denoted by $\zeta\zeta'_p := \zeta_p+\zeta'_p$.

For example, the following figure shows the tensor product of two leg bundles.

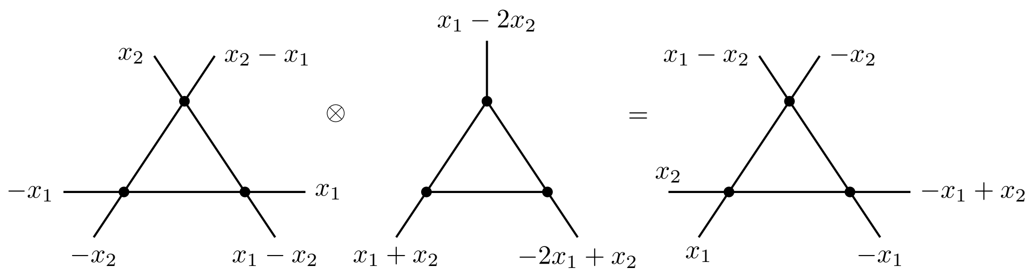


Figure 5. Geometrically, this shows the relation $\tau_{\mathbb{C}P^2}^*\otimes(\gamma^{\otimes 3}\otimes\epsilon_{1,1})\simeq\tau_{\mathbb{C}P^2}$.

### 3.3. Splitting into the line bundle from GKM theoretical viewpoint.

The following notion is important to state our main result.

Definition 3.4 (Splitting). For a rank $r$ leg bundle $\xi$ over a GKM graph $\Gamma=(G,\alpha,\nabla)$, if there is a connection $\widehat{\sigma}^{\xi}$ (might be different from the original connection $\sigma^{\xi}$) on $\xi$ such that $\xi$ with the connection $\widehat{\sigma}^{\xi}$ is the Whitney sum of line bundles $\gamma_1\oplus\cdots\oplus\gamma_r$, then we call $\xi$ a *splitting bundle*.

For example, the decomposition

$$\tau_{\mathbb{C}P^2}\oplus\epsilon_{(0,0)}=\gamma\oplus(\gamma\otimes\epsilon_{1,0})\oplus(\gamma\otimes\epsilon_{0,1}).$$

can be computed by the following decomposition of the leg bundles:

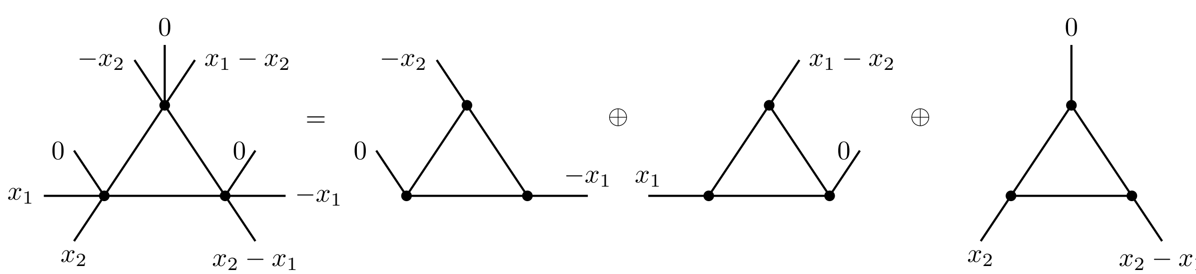


Figure 6. Splitting of the rank 3 leg bundle into the ling bundles.

## 4. Main theorem

In this section, we state the main theorem of this paper.

**4.1. Main theorem.** Let $\Gamma = (G, \alpha, \nabla)$ be a $(2,2)$-type GKM graph for $G = (\mathcal{V}, \mathcal{E})$ and $\xi$ be a rank 2 leg bundle over $\Gamma$. Then, we may write

- $\mathcal{V} = \{p_1, \ldots, p_r\}$ and $\mathcal{E} = \{e_1, \ldots, e_r\}$, where $e_i$ is the edge connecting $p_i$ and $p_{i+1}$ for $i = 1, \ldots, r-1$ and $e_r$ is the edge connecting $p_r$ and $p_1$, i.e., $G$ is the boundary of the $r$-gon for some $r \geq 2$;
- the fiber over $p \in \mathcal{V}$ is $[2]_p := \{\ell_{p,1}, \ell_{p,2}\}$,

where in the above notations $\ell_{p,1} = (p, 1)$ and $\ell_{p,2} = (p, 2)$.

Then the following lemma is straightforward:

LEMMA 4.1. *For any label on $\xi$, the connection on $\xi$ is one of the following or both of them:*

(1) $\sigma^{\xi}_{e_r} \circ \cdots \circ \sigma^{\xi}_{e_1}(\ell_{p_1,i}) = \ell_{p_1,i}$ *for* $i = 1, 2$*, i.e.,* $\xi$ *is the splitting bundle (in fact, it splits into two line bundles);*
(2) $\sigma^{\xi}_{e_r} \circ \cdots \circ \sigma^{\xi}_{e_1}(\ell_{p_1,1}) = \ell_{p_1,2}$*, i.e.,* $\xi$ *is not the splitting bundle.*

By changing the order of legs, for the 1st case in Lemma 4.1, we may assume that $\sigma^{\xi}_{e_i}$ for all $i = 1, \ldots, r$ is identity. In this case, we say that $\xi$ admits a *splitting type* connection. Similarly, for the 2nd case in Lemma 4.1, we may assume that only one connection on an edge is twisting and the others are identity; we say that $\xi$ admits a *non-splitting type* connection.

The following figure shows that the rank 2 leg bundle $\xi$ over the GKM graph induced from $\mathbb{C}P^2$ which admit both of the splitting type and the non-splitting type connections.

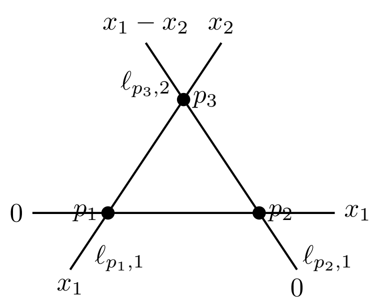


FIGURE 7. Rank 2 leg bundle with two connections.

Now we may state the main theorem:

THEOREM 4.2. *Let $\xi$ be a rank 2 vector bundle over a $(2,2)$-type GKM graph $\Gamma$. Then the following two conditions are equivalent:*

(1) $\xi$ *admits both of the splitting type and the non-splitting type connections;*
(2) *The projectivization* $\Pi(\xi)$ *is not a GKM graph.*

The following figure shows that the projectivization (for the non-splitting connection) of the leg bundle in Figure 7. This does not satify the 2-independency aroung the vertex $\ell_{p_1,1}$; therefore, this is not the GKM graph.

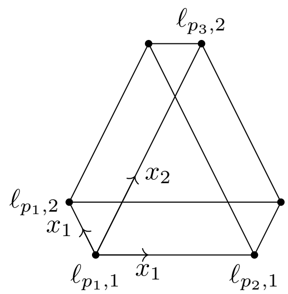


FIGURE 8. The projectivization $\Pi(\xi)$ of the leg bundle $\xi$ with the non-splitting connection in Figure 7.

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