Equivariant cohomology of complex quadrics from a combinatorial point of view

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1. Introduction

This article is the research announcement of the paper [**Ku**] about the computation of the torus equivariant cohomology of the complex quadrics by GKM theory.

1.1. Basic properties of the complex quadrics. The *complex quadrics* Q_m is the following space defined by the quadratic equations:

$$Q_m := \left\{ [z_1 : \cdots : z_{m+2}] \in \mathbb{C}P^{m+1} \;\middle|\; \sum_{i=1}^{m+2} z_i^2 = 0 \right\}.$$

We first recall some properties of this space.

Since this space is the solutions of the equation $\sum_{i=1}^{m+2} z_i^2 = 0$ in $\mathbb{C}P^{m+1}$, its dimension satisfies that $\dim Q_m = 2m$. Moreover, the equation $\sum_{i=1}^{m+2} z_i^2 = 0$ regards as the (standard Euclidean) inner product $\langle \mathbf{z}, \mathbf{z} \rangle = 0$ for $\mathbf{z} = (z_1, \ldots, z_{m+2})$. So there is the transitive $SO(m+2)$-action on Q_m by the standard multiplication. By computing the isotropy subgroup of the point $[0 : \cdots : 0 : 1 : \sqrt{-1}] \in Q_m$, there is a diffeomorphism onto the following homogeneous space:

$$Q_m \simeq SO(m+2)/SO(m) \times SO(2).$$

This structure shows that the maximal torus of $SO(m+2)$ acts on Q_m, i.e., T^{n+1} acts on Q_{2n+1} and Q_{2n} respectively. Note that the T^{n+1}-action on Q_{2n} defined by this way is non-effective because the maximal torus T^{n+1} in $SO(2n+2)$ has the non-trivial center $\mathbb{Z}_2 = \{\pm I_{2n+2}\}$.

1.2. The cohomology ring and the main theorem of this paper. The cohomology ring of Q_m over the integer coefficient has the following ring structure (see [**La72, La74**] for $H^*(Q_{2n})$ or [**EKM08**, Excercise 68.3] for $H^*(Q_m)$ as the Chow ring[1]):

$$H^*(Q_m) \simeq \begin{cases} \mathbb{Z}[c,x]/\langle c^{n+1} - 2x, x^2 \rangle & \text{if } m = 2n+1, \quad \text{where } \deg c = 2,\ \deg x = 2n+2 \\ \mathbb{Z}[c,x]/\langle c^{2n+1} - 2cx, x^2 - c^{2n}x \rangle & \text{if } m = 4n, \quad \text{where } \deg c = 2,\ \deg x = 4n \\ \mathbb{Z}[c,x]/\langle c^{2n+2} - 2cx, x^2 \rangle & \text{if } m = 4n+2, \quad \text{where } \deg c = 2,\ \deg x = 4n+2 \end{cases}$$

Note that the ring structure of $H^*(Q_{2n})$ depends on whether n is even or odd.

The purpose of this paper is to understand the difference of ring structures of $H^*(Q_{2n})$ from GKM theory, i.e., we describe the difference between $H^*(Q_{4n})$ and $H^*(Q_{4n+2})$ by using the combinatorics of graphs. In order to do that, we first compute the GKM graph $\mathcal{Q}_{2n}$ of the effective T^{n+1}-action on Q_{2n} in Section 2. Note that by the ring structure of $H^*(Q_{2n})$ as above, we have $H^{odd}(Q_{2n}) = 0$, i.e., Q_{2n} is *equivariantly formal*. Therefore, by using GKM theroy (see [**GKM98, GZ01**]), the equivariant cohomology $H^*_{T^{n+1}}(Q_{2n})$ is isomorphic to the *graph equivariant cohomology* $H^*(\mathcal{Q}_{2n})$ of the GKM graph $\mathcal{Q}_{2n}$ (see Section 3). The main theorem of this paper is to show the ring structure of $H^*(\mathcal{Q}_{2n})$ by generators and relations. As a consequence of the main

[1] Since Q_m also can be regarded as the homogeneous space of the affine algebraic group $SO(m+2, \mathbb{C})$, it follows from [**EH13**, Appendix C.3.4] that its Chow ring is isomorphic to its cohomology ring, i.e., $A^*(Q_m) \simeq H^{2*}(Q_m; \mathbb{Z})$.

theorem in Section 3 (see Lemma 3.1 and Theorem 3.12), we have the following ring structure of the equivariant cohomology of Q_{2n} with T^{n+1}-action (the notations will intrduce in Section 3)

THEOREM 1.1. *There exists the following isomorphism as a ring:*

$$H^*_{T^{n+1}}(Q_{2n}) \simeq \mathbb{Z}[\mathcal{Q}_{2n}].$$

In particular, the generators of $\mathbb{Z}[\mathcal{Q}_{2n}]$ are given by (generalized) GKM subgraphs of $\mathcal{Q}_{2n}$. This gives the unified formula of the ring structures of $H^*_{T^{2n+1}}(Q_{4n})$ and $H^*_{T^{2n+2}}(Q_{4n+2})$. We finally describe that the difference between the ring structures of $H^*(Q_{4n})$ and $H^*(Q_{4n+2})$ by using the relations in $\mathbb{Z}[\mathcal{Q}_{2n}]$ (see Section 4).

2. The GKM graph of the effective T^{n+1}-action on Q_{2n}

In this section, we compute the GKM graph of the T^{n+1}-action on Q_{2n}. The basic facts of the GKM graph (including the definition) refer to [**GZ01, Ku09**].

2.1. The T^{n+1}-action on Q_{2n} which preserves the complex structure. Recall that the (even degree) complex quadrics Q_{2n} is diffeomorphic to the following space of solutions of the quadric equation (see [**Se06**, Chapter V.1, 1.1 Theorem.]).

$$Q_{2n} := \left\{ [z_1 : \cdots : z_{2n+2}] \in \mathbb{C}P^{2n+1} \;\middle|\; \sum_{i=1}^{n+1} z_i z_{2n+3-i} = 0 \right\}.$$

For this space, there exists the natural T^{n+1}-action on Q_{2n} defined by

$$[z_1 : \cdots : z_{2n+2}] \mapsto [z_1t_1 : z_2t_2 : \cdots : z_{n+1}t_{n+1} : t_{n+1}^{-1}z_{n+2} : \cdots t_2^{-1}z_{2n+1} : t_1^{-1}z_{2n+2}], \tag{2.1}$$

where $(t_1, \ldots, t_{n+1}) \in T^{n+1}$. This is equivariantly diffeomorphic to Q_{2n} with $T^{n+1} \subset SO(2n+2)$ action in Section 1, and this action also has the finite kernel $\mathbb{Z}_2$ which is the center of T^{n+1} in $SO(2n+2)$. So if we divide T^{n+1} by $\mathbb{Z}_2$, then we obtain the effective T^{n+1}-action on Q_{2n}. It is easy to check that this T^{n+1}-action preserves the complex structure on $\mathbb{C}P^{2n+1}$, because this T^{n+1}-action is induced from the representation of $T^{n+1} \to GL(2n+2, \mathbb{C})$ and $GL(2n+2, \mathbb{C})$ action on $\mathbb{C}P^{2n+1}$ preserves the complex structure on $\mathbb{C}P^{2n+1} := (\mathbb{C}^{2n+2} \setminus \{0\})/\mathbb{C}^*$.

Henceforth, the notation Q_{2n} represents the space defined as above with the T^{n+1}-action defined in (2.1).

2.2. GKM graph of the T^{n+1}-action on Q_{2n}. By definition, the GKM graph consists of the fixed points (vertices) and the invariant 2-spheres (edges), and the labels on edges (the *axial function* of the GKM graph) which are defined by the tangential representations on fixed points. In this section, we compute them for the T^{n+1}-action (2.1) on Q_{2n}.

Because of (2.1), the fixed points of Q_{2n} are

$$Q_{2n}^T = \{[e_i] \mid i = 1, \ldots, 2n+2\},$$

where $[e_i] = [0 : \cdots : 0 : 1 : 0 : \cdots : 0] \in \mathbb{C}P^{2n+1}$ (only ith coordinate is 1). Moreover, the invariant $S^2(\simeq \mathbb{C}P^1)$'s are

$$[z_i : z_j] := [0 : \cdots : 0 : z_i : 0 : \cdots : 0 : z_j : 0 \cdots : 0] \in Q_{2n+2} \tag{2.2}$$

where $i + j \neq 2n+3$. Therefore, we can define the graph $\Gamma_{2n} := (V_{2n}, E_{2n})$ from the T^{n+1}-action on Q_{2n} as follows (also see Figure 1):

- the set of vertices $V_{2n} = [2n+2] := \{1, \ldots, 2n+2\}$;
- the set of edges $E_{2n} = \{ij \mid i, j \in [2n+2] \text{ such that } i \neq j,\ i + j \neq 2n+3\}$.

REMARK 2.1. For convenience, we often denote the vertex $j \in V_{2n}$ such that $i + j = 2n+3$ by $\bar{i}$. Namely,

$$V_{2n} = [2n+2] = \{1, 2, \ldots, n+1, \overline{n+1}, \overline{n}, \ldots, \overline{1}\}.$$

By this notation, the set of edges can be written by

$$E_{2n} = \{ij \mid i, j \in V_{2n} \text{ such that } j \neq i, \bar{i}\}$$

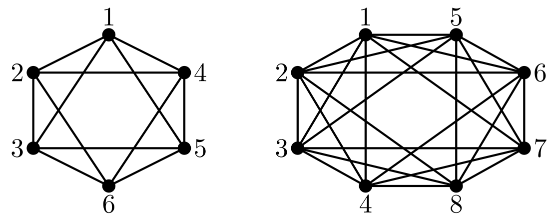

FIGURE 1. The graph induced from the T^3-action on Q_4 (left) and the T^4-action on Q_6 (right).

We next compute the tangential representations around fixed points and put the label on edges, called an *axial function* on edges and denoted by $\alpha : E_{2n} \to H^2(BT^{n+1})$. Recall that the tangential representations around the fixed points decompose into the complex 1-dimensional irreducible representations. Each complex 1-dimensional irreducible representation corresponds to the tangential representation on the fixed point of the invariant 2-sphere. So it is enough to compute the tangential representation on each invariant 2-sphere $[z_i : z_j] \in Q_{2n}$ (see (2.2)). By the definition of T^{n+1}-action on $[z_i : z_j]$, we may write the action $t = (t_1, \dots, t_{n+1}) \in T^{n+1}$ on $[z_i : z_j]$ by

$$[z_i : z_j] \mapsto [p_i(t)z_i : p_j(t)z_j],$$

where $p_i : T^{n+1} \to S^1$ is the surjective homomorphism defined by

$$p_i(t) = \begin{cases} t_i & \text{if } i \in [n+1] \\ t_{\bar{i}}^{-1} & \text{if } i \in \{n+2, \dots, 2n+2\} \end{cases}$$

Therefore, the axial function $\alpha : E_{2n} \to H^2(BT^{n+1})$ is defined by the following equation (see Figure 2):

$$\alpha(ij) = x_j - x_i, \tag{2.3}$$

where $x_i \in H^2(BT^{n+1}) \simeq (\mathfrak{t}_{\mathbb{Z}}^{n+1})^* \simeq \mathrm{Hom}(T^{n+1}, S^1)$ is the element corresponding to $p_i \in \mathrm{Hom}(T^{n+1}, S^1)$ defined by

- $x_i = p_i$ for $i \in [n+1]$;
- $x_i = -p_i = -x_{\bar{i}}$ for $i \in \{n+2, \dots, 2n+2\}$.

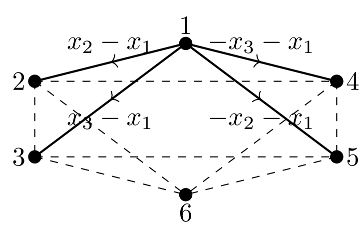

FIGURE 2. The axial function around the vertex 1 of the GKM graph induced from the T^3-action on Q_4. Note that $\overline{6} = 1$, $\overline{5} = 2$, $\overline{4} = 3$.

2.3. GKM graph of the effective T^{n+1}-action. Since the T^{n+1}-action (2.1) on Q_{2n} is not effective, the axial function defined by (2.3) does not satisfy the effectiveness conditions. For example, around the vertex $1 \in V_{2n}$, the axial functions are

$$x_2 - x_1, \dots, x_{n+1} - x_1, -x_{n+1} - x_1, \dots, -x_2 - x_1 \in (\mathfrak{t}_{\mathbb{Z}}^{n+1})^*, \tag{2.4}$$

and it is easy to check that these vectors are not primitive because any $n+1$ vectors do not span $(\mathfrak{t}_{\mathbb{Z}}^{n+1})^*$, e.g., $x_2 - x_1, \ldots, x_{n+1} - x_1$ and $-x_{n+1} - x_1$, i.e., the effectiveness condition does not hold. Therefore, we can not use the usual GKM theory directly[2].

However, if we replace the labels with primitive vectors, then we can get the axial function defined from the effective $T^{n+1}(\simeq T^{n+1}/\mathbb{Z}_2)$-action on Q_{2n}, where $\mathbb{Z}_2 = \{\pm 1\}$ is the kernel of the non-effective T^{n+1}-action in (2.1). For example, we replace vectors (2.4) with the following vectors (respectively)

$$x_1,\ \ldots,\ x_{n+1},\ -x_{n-1} + x_n + x_{n+1},\ \ldots,\ -x_1 + x_n + x_{n+1}.$$

Then, these vectors are primitive. Moreover, by using the connection on the GKM graph, other axial functions are automatically determined. Therefore, we may define the axial function as follows (also see Remark 2.3 and Figure 3):

Definition 2.2. Set $f : V_{2n} \to H^2(BT^{n+1})$ as

$$f(j) = \begin{cases} x_{j-1} - x_{n+1} & j = 1, \ldots, n+2 \\ x_n - x_{2n+2-j} & j = n+3, \ldots, 2n+2 \end{cases}$$

where $x_0 = 0$ and $\langle x_1, \ldots, x_{n+1} \rangle = H^2(BT^{n+1})$[3]. Then the axial function $\alpha : E_{2n} \to H^2(BT^{n+1})$ is defined by

$$\alpha(ij) := f(j) - f(i)$$

for $j \neq i, \bar{i}$.

We denote the GKM graph (Γ_{2n}, α) (or equivalently (Γ_{2n}, f), called a 0-*cochain presentation*) for $\Gamma = (V_{2n}, E_{2n})$ defined in Definition 2.2 by $\mathcal{Q}_{2n}$.

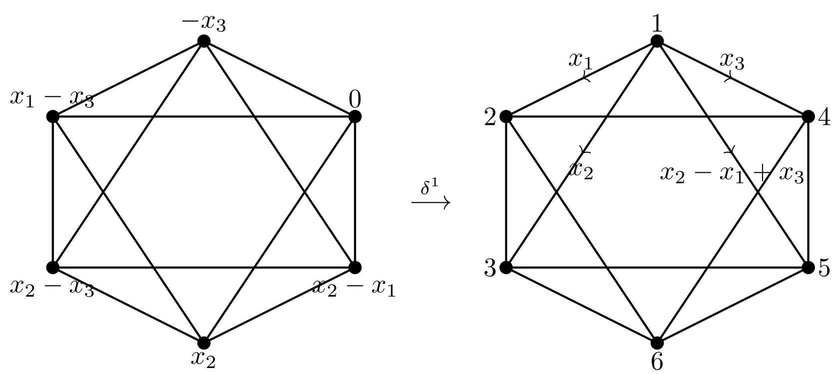

Figure 3. The GKM graph $\mathcal{Q}_{2n}$ when $n = 2$. The right figure shows that the axial function $\alpha : E_4 \to H^2(BT^3)$ of $\mathcal{Q}_4$ around the vertex 1. The left figure shows its 0-cochain presentation $f : V_4 \to H^2(BT^3)$.

2.4. Remarks from the sheaves on graphs. Due to [**BM01**], we can define the *structure sheaf* (or the *sheaf of rings*) over the graph Γ (with an appropriate topology) from the GKM graph (Γ, α), say $\mathcal{M}$ (also see [**Ku16**]), whose global sections are isomorphic to the graph equivariant cohomology, i.e., $H^0(\Gamma; \mathcal{M}) \simeq H^*(\Gamma, \alpha)$ (see Section 3). On the other hand, by using [**Ha21**], we may also regard the axial function $\alpha : E_{2n} \to H^2(BT^{n+1})$ as the element of the 1-cochain of the structure sheaf (in the sense of [**BM01**]), i.e.,

$$C^1(\Gamma_{2n}; \mathcal{M}) := \bigoplus_{e \in E_{2n}} H^*(BT^{n+1}).$$

[2]More precisely, this means that the graph equivariant cohomology (see Section 3) is not isomorphic to the equivariant cohomology over integer coefficient (also see [**KKLS20**, Remark 4.5]). To apply the GKM theory for the non-effective torus action, we need to modify the definition of the graph equivariant cohomology (see Appendix A).

[3]More precisely, $H^2(BT^{n+1})$ in Definition 2.2 is $H^2(B(T^{n+1}/\mathbb{Z}_2))$ by identifying them as $T^{n+1}/\mathbb{Z}_2 \simeq T^{n+1}$.

On the other hand, the map $f : V_{2n} \to H^2(BT^{n+1})$ in Definition 2.2 is the element of the 0-cochain of the structure sheaf, i.e.,

$$C^0(\Gamma_{2n}; \mathcal{M}) := \bigoplus_{p \in V_{2n}} H^*(BT^{n+1}).$$

The axial function defined in Definition 2.2 is nothing but the image of the connection homomorphism

$$\delta^1 : C^0(\Gamma_{2n}; \mathcal{M}) \to C^1(\Gamma_{2n}; \mathcal{M})$$

which is defined by $\delta^1(f)(e) := f(q) - f(p)$ for $f \in C^0(\Gamma_{2n}; \mathcal{M})$ and the oriented edge $e = pq$. Namely, there is the following relation between the axial function α and the 0-cochain presentation f:

$$\delta^1(f) = \alpha.$$

This is the reason why we call f in Definition 2.2 a 0-*cochain presentation* of the axial function α (also see [**KM**]).

Remark 2.3. There are several choices of 0-cochain presentations because every elements in $(\delta^1)^{-1}(\alpha)$ can be a 0-cochain presentation. However, using a 0-cochain presentation f is much simpler to draw figures (see Figure 3) than using the axial function α. So in this paper, we fix one of the 0-cochain presentations as in Definition 2.2 instead of using the axial function.

Remark 2.4. Let (Γ, α) be a GKM graph and $\mathcal{M}$ be its structure sheaf in [**BM01**]. Then we may define the following sheaf cohomologies (see [**Ha21**]):

$$H^0(\Gamma; \mathcal{M}) := \mathrm{Ker}(\delta^1) \simeq H^*(\Gamma, \alpha);$$
$$H^1(\Gamma; \mathcal{M}) := C^1(\Gamma; \mathcal{M})/\mathrm{Im}(\delta^1).$$

By Remark 2.3, it is easy to check that there exists a 0-cochain presentation f if and only if $\alpha \in \mathrm{Im}(\delta^1)$, i.e.,

$$[\alpha] = 0 \in H^1(\Gamma; \mathcal{M}).$$

Therefore, if $H^1(\Gamma; \mathcal{M}) = 0$, then the axial function which defines $\mathcal{M}$ has a 0-cochain presentation.

Remark 2.5. There is an example that does not have any 0-cochain presentations of the axial function α. By easy computations, we can not take any 0-cochain presentation of the axial function of the torus graph defined from the standard T^2-action on S^4 (see e.g. [**MMP07**]). This implies that the axial function $[\alpha] \in H^1(\Gamma; \mathcal{M})$ is a non-zero class for the structure sheaf of the torus graph of the T^2-action on S^4. More generally, if there is a multi-edge in the GKM graph, then we cannot take a 0-cochain presentation of the axial function α.

3. Graph equivariant cohomology $H^*(\mathcal{Q}_{2n})$ and equivariant cohomology $H^*_{T^{n+1}}(Q_{2n})$

The *graph equivariant cohomology* of the GKM graph $\mathcal{Q}_{2n}$ is defined by

$$H^*(\mathcal{Q}_{2n}) := \{h : V_{2n} \to H^*(BT^{n+1}) \mid h(i) - h(j) \equiv 0 \mod \alpha(ij) \text{ for } ij \in E_{2n}\}. \tag{3.1}$$

Because $H^{odd}(Q_{2n}) = 0$, it follows from [**GKM98, FP07**] that we have the following lemma:

Lemma 3.1. *For the effective T^{n+1}-action on Q_{2n}, the following isomorphism holds:*

$$H^*_{T^{n+1}}(Q_{2n}) \simeq H^*(\mathcal{Q}_{2n}).$$

So to compute the equivariant cohomology of Q_{2n}, it is enough to compute the graph equivariant cohomology $H^*(\mathcal{Q}_{2n})$. In this section, we introduce the generators and relations of $H^*(\mathcal{Q}_{2n})$.

3.1. Degree 2 generators. We first define the degree 2 generators, i.e., we will define the generators in $H^2(\mathcal{Q}_{2n})$.

Definition 3.2 (degree 2 generators). Let $I \subset V_{2n}$ be the set $I = V_{2n} \setminus \{i\}$ for some $i \in V_{2n} = [2n+2]$. Define $M_I : V_{2n} \to H^2(BT^{n+1})$ by

$$M_I(j) = \begin{cases} \alpha(ji) = f(i) - f(j) & j \neq \bar{i} \\ \alpha(\bar{i}k) + \alpha(\bar{i}\bar{k}) = f(k) + f(\bar{k}) - 2f(\bar{i}) & j = \bar{i} \end{cases}$$

where k can be taken any $k \in V_{2n} \setminus \{i, \bar{i}\}$.

Notice that the following proposition holds for the axial function on $\mathcal{Q}_{2n}$:

Proposition 3.3. For every $j, k \in V_{2n} \setminus \{i, \bar{i}\}$, the following equation holds:

$$\alpha(\bar{i}j) + \alpha(\bar{i}\bar{j}) = \alpha(\bar{i}k) + \alpha(\bar{i}\bar{k})$$

It follows from Proposition 3.3 that Definition 3.2 is well-defined.
By checking $M_I(j) - M_I(k) \equiv 0 \mod \alpha(jk)$ for every $jk \in E_{2n}$, we have the following lemma.

Lemma 3.4. *For every $i \in V_{2n}$, $M_I \in H^2(\Gamma, \alpha)$, where $I = V_{2n} \setminus \{i\}$.*

Example 3.5. For $I = V_4 \setminus \{6\} = V_4 \setminus \{\bar{1}\}$, Figure 4 shows the class $M_I \in H^2(\mathcal{Q}_4)$.

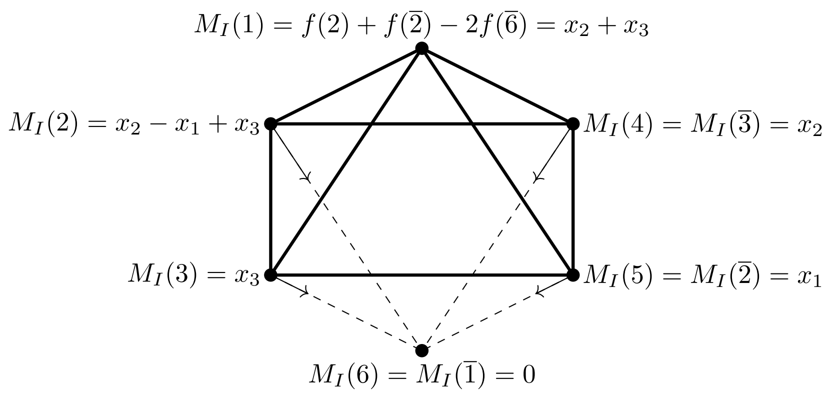

Figure 4. M_I for $I = V_4 \setminus \{\bar{1}\}$.

Note that $M_I(j)$ for $j \neq \bar{i}$ is the normal axial function $\alpha(ji)$ of j of the full-subgraph $I \subset V_{2n}$[4].

3.2. Higher degree generators. We next define the degree $2k$ generators, i.e., we will define the generators in $H^{2k}(\mathcal{Q}_{2n})$ for $k \geq n$.

Definition 3.6 (degree $(\geq)2n$ generators). Let $K \subset V_{2n} = [2n+2]$ be a subset that satisfies if $i \in K$, then $\bar{i} \notin K$ (or equivalently $\{i, \bar{i}\} \not\subset K$ for all $i \in V_{2n}$). Define $\Delta_K : V_{2n} \to H^{4n-2(|K|-1)}(BT^{n+1})$ by

$$\Delta_K(j) = \begin{cases} \prod_{k \notin K \cup \{\bar{j}\}} \alpha(jk) = \prod_{k \notin K \cup \{\bar{j}\}} (f(k) - f(j)) & j \in K \\ 0 & j \notin K \end{cases}$$

The following lemma is straightforward.

Lemma 3.7. *Let $|K|$ be the cardinality of the finite set K. Then $\Delta_K \in H^{4n-2(|K|-1)}(\Gamma, \alpha)$.*

[4] It is easy to check that such class in $H^2(\mathcal{Q}_{2n})$ is unique, that is, $M_I(\bar{i})$ is automatically determined (also see Section 5).

Remark 3.8. By the definition of edges E_{2n}, every pair $\{i, j\}$ of vertices in K are connected by an edge $ij \in E_{2n}$. Therefore, the full subgraph of K consists of a complete subgraph in $\mathcal{Q}_{2n}$. Note that the generator Δ_K is nothing but the Thom class of the GKM subgraph whose vertices consist of K (see [**MMP07**]).

Geometrically, Δ_K is the equivariant Thom class of the projective space in Q_{2n} whose fixed points consist of K. For example, there exists the following subspace in Q_{2n}:

$$\{[z_1 : z_2 : \cdots : z_{n+1} : 0 : \cdots : 0] \in Q_{2n} \mid z_i \in \mathbb{C}\} \simeq \mathbb{C}P^n.$$

Then, for $K = [n+1]$, the generator Δ_K is the equivariant Thom class of this $\mathbb{C}P^n$ (Figure 5 shows this class when $n = 2$).

Example 3.9. For the GKM graph $\mathcal{Q}_4$, the set of vertices $K = \{1, 2, 3\}$ satisfies the condition which defines Δ_K. Figure 5 shows the generator $\Delta_K \in H^4(\mathcal{Q}_4)$.

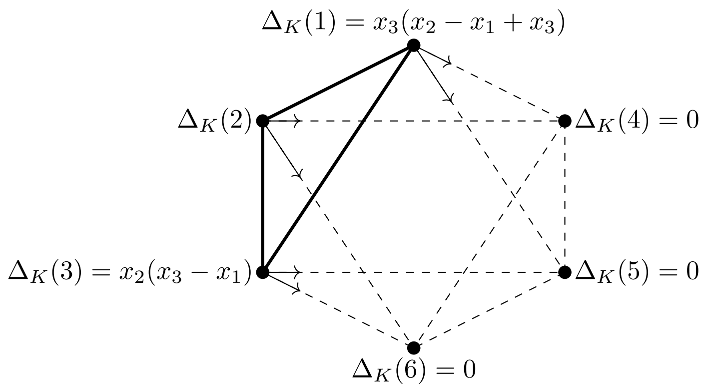

Figure 5. Δ_K for $K = \{1, 2, 3\}$, where $\Delta_K(2) = (x_3 - x_1)(x_2 - x_1 + x_3)$.

Example 3.10. For the GKM graph $\mathcal{Q}_4$, the set of vertices $L = \{1, 2\}$ also satisfies the condition which defines Δ_L. Figure 6 shows the generator $\Delta_L \in H^6(\mathcal{Q}_4)$.

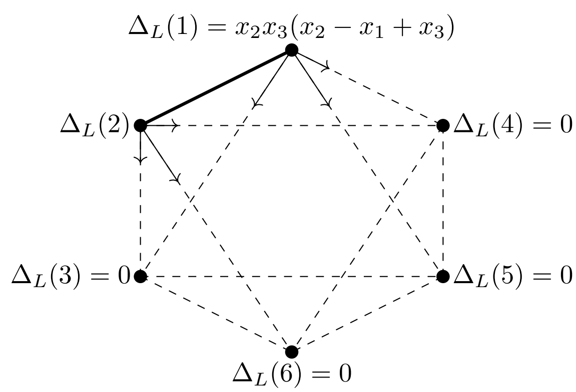

Figure 6. Δ_L for $L = \{1, 2\}$, where $\Delta_L(2) = (x_2 - x_1)(x_3 - x_1)(x_2 - x_1 + x_3)$

3.3. Relations among generators.

We next introduce five relations among M_I's and Δ_K's.

Relation 1. We define the following elements for $J \subset V_{2n}$:

$$G_J := \begin{cases} M_J & \text{if } J = V_{2n} \setminus \{i\} \text{ for some } i \in V_{2n} \\ \Delta_J & \text{if } J \text{ satisfies that } \{i, \bar{i}\} \not\subset J \text{ for every } i \in V_{2n} \end{cases}$$

Then, the following relation holds:

$$\prod_{\cap J=\emptyset} G_J = 0. \tag{3.2}$$

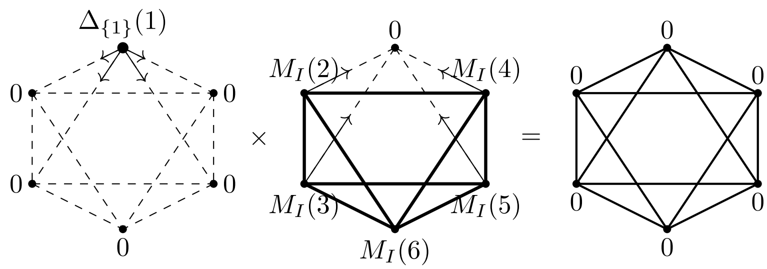

FIGURE 7. Figure of Relation 1. This represents the relation $\Delta_{\{1\}} \cdot M_I = 0$ for $I = V_4 \setminus \{1\}$.

RELATION 2. We define the element $X \in H^2(\mathcal{Q}_{2n})$ as the map $X : V_{2n} \to H^2(BT^{n+1})$ defined by

$$X(k) := \alpha(kj) + \alpha(k\overline{j}),$$

for all $k \in V_{2n} \setminus \{j, \overline{j}\}$, where $j \in V_{2n}$ can be taken any element if $j \neq k, \overline{k}$ (by Proposition 3.3).

Let $i \in V_{2n}$, $I = V_{2n} \setminus \{i\}$ and $\overline{I} = V_{2n} \setminus \{\overline{i}\}$. Then, the following relation holds:

$$M_I + M_{\overline{I}} = X. \tag{3.3}$$

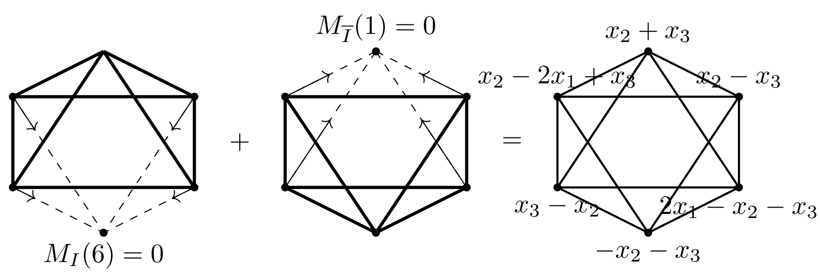

FIGURE 8. Figure of Relation 2. This represents the relation $M_I + M_{\overline{I}} = X$ where $I = V_4 \setminus \{6\}$ and $\overline{I} = V_4 \setminus \{1\}$.

RELATION 3. Assume that the subset $I \subset V_{2n}$ satisfies that $|I| = n$ and there exists the unique pair $\{a, \overline{a}\} \subset I^c$. By using the pigeonhole principle, in this case $K = (I \cup \{a\})^c$ and $L = (I \cup \{\overline{a}\})^c$ satisfy the condition which can define the generators $\Delta_K, \Delta_L \in H^{2n}(\mathcal{Q}_{2n})$. Then, the following relation holds:

$$\prod_{i \in I} M_{V_{2n} \setminus \{i\}} = \Delta_{(I \cup \{a\})^c} + \Delta_{(I \cup \{\overline{a}\})^c}. \tag{3.4}$$

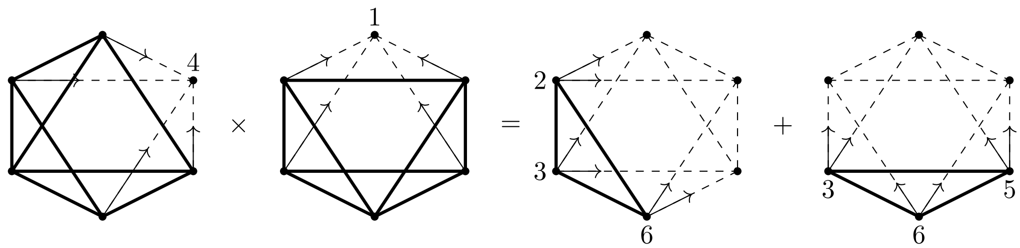

FIGURE 9. Figure of Relation 3. This represents the following relation:

$$M_{V_4\setminus\{4\}} \cdot M_{V_4\setminus\{1\}} = \Delta_{\{2,3,6\}} + \Delta_{\{3,5,6\}},$$

where $I = \{1,4\} \subset V_4$ (for $n = 2$). Note that in this case $I^c = \{2,3,5,6\}$ and $a = 2, \overline{a} = 5$. Moreover, $(I \cup \{5\})^c = \{2,3,6\}$ and $(I \cup \{2\})^c = \{3,5,6\}$.

RELATION 4. Let $I = V_{2n} \setminus \{i\}$ for some $i \in V_{2n}$ and $K \subset V_{2n}$ be a subset which can define the generator Δ_K. Assume that $K \not\subset I$ and $K \cap I \neq \emptyset$ (equivalently $\{i\} \subsetneq K$). Then, the following relation holds:

$$\Delta_K \cdot M_I = \Delta_{K\cap I}. \tag{3.5}$$

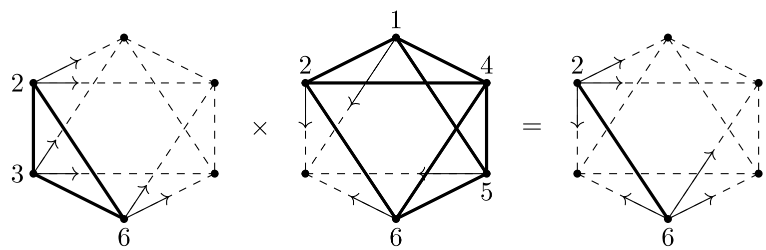

FIGURE 10. Figure of Relation 4. This represents the relation $\Delta_{\{2,3,6\}} \cdot M_{V_4\setminus\{3\}} = \Delta_{\{2,6\}}$.

RELATION 5. Let $K, H \subset V_{2n}$ be subsets with $|K| = |H| = n+1$ which define $\Delta_K, \Delta_H \in H^{2n}(\mathcal{Q}_{2n})$. Then, the following relation holds:

$$\Delta_K \cdot \Delta_H = \Delta_{K\cap H} \cdot \left(\sum_{i=0}^{|K\cap H|-1} (-1)^i X^i \cdot \sigma_{|K\cap H|-1-i}(M_I \mid K \cup H \subset I) \right), \tag{3.6}$$

where $X \in H^2(\mathcal{Q}_{2n})$ is the element defined in Relation 2 and σ_j is the symmetric polynomial with degree j.

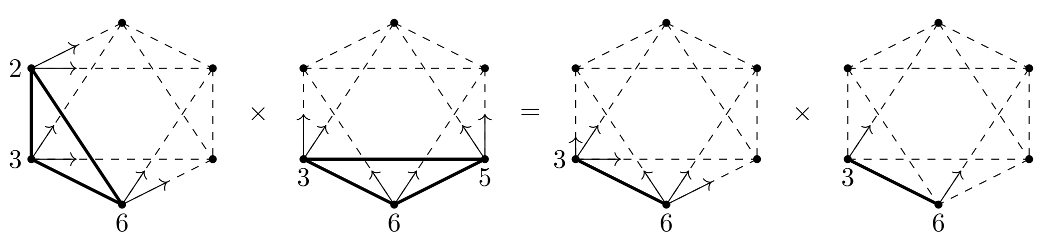

FIGURE 11. Figure of Relation 5 (also see Figure 12), where $K = \{2,3,6\}$, $H = \{3,5,6\}$. This represents the following relation:

$$\begin{aligned}\Delta_{\{2,3,6\}} \cdot \Delta_{\{3,5,6\}} =& \Delta_{\{3,6\}} \cdot (\sigma_1(M_{V_4\setminus\{1\}}, M_{V_4\setminus\{4\}}) - X)\\ =& \Delta_{\{3,6\}} \cdot (M_{V_4\setminus\{4\}} + M_{V_4\setminus\{1\}} - X),\end{aligned}$$

because $K \cap H = \{3,6\}$ and $K \cup H = \{2,3,5,6\} \subset I$ (so $I = V_4 \setminus \{1\}$ or $V_4 \setminus \{4\}$).

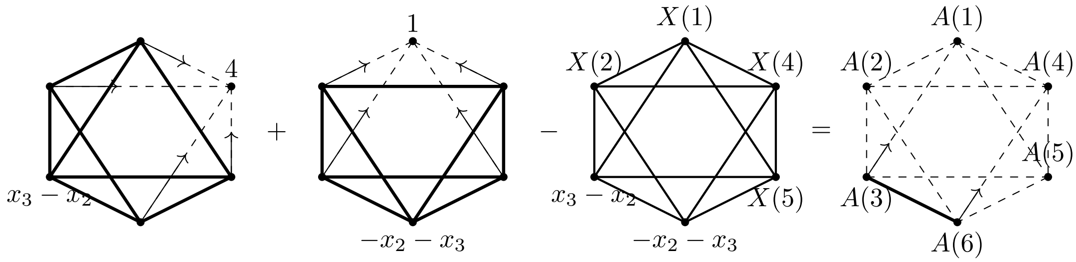

FIGURE 12. Figure about the element $A = M_{V_4\setminus\{4\}} + M_{V_4\setminus\{1\}} - X$ in Figure 11. Note that $A(3) = A(6) = -x_2$ by Figure 3. Moreover, $A(1), A(2), A(4), A(5)$ might not be $0 \in H^2(BT^3)$; however, $\Delta_{\{3,6\}}(1) = \Delta_{\{3,6\}}(2) = \Delta_{\{3,6\}}(4) = \Delta_{\{3,6\}}(5) = 0$.

3.4. Main theorem. Now we may state the main theorem of this paper. To do that, we first define the following notations.

DEFINITION 3.11. Set

$$\mathcal{M} := \{M_I \mid I = V_{2n} \setminus \{i\} \text{ for } i \in V_{2n}\}, \quad \mathcal{D} := \{\Delta_K \mid K \subset V_{2n} \text{ such that if } i \in K \text{ then } \bar{i} \notin K\}.$$

Put the polynomial ring generated by $\mathcal{M}, \mathcal{D}$ by

$$\mathbb{Z}[\mathcal{M}, \mathcal{D}].$$

Let $\mathcal{I}$ be the ideal in $\mathbb{Z}[\mathcal{M}, \mathcal{D}]$ generated by the 5 relations defined in Section 3.3. Then, we define

$$\mathbb{Z}[\mathcal{Q}_{2n}] := \mathbb{Z}[\mathcal{M}, \mathcal{D}]/\mathcal{I}.$$

The following theorem is the main theorem of this paper.

THEOREM 3.12. *There is the following isomorphism:*

$$\mathbb{Z}[\mathcal{Q}_{2n}] \simeq H^*(\mathcal{Q}_{2n}).$$

We prove this theorem in [**Ku**].
Together with Lemma 3.1, we have Theorem 1.1.

4. Combinatorial interpretation of the difference between $H^*(Q_{4n})$ and $H^*(Q_{4n+2})$

In this section, we first compute the ordinary cohomology from Theorem 1.1 and consider the meaning of the difference between $H^*(Q_{4n})$ and $H^*(Q_{4n+2})$ from a combinatorial point of view.

4.1. Ordinary cohomology $H^*(Q_{2n})$. Let $H^*(BT^{n+1}) = \mathbb{Z}[x_1, \ldots, x_{n+1}]$. The elements $x_1, \ldots, x_{n+1}$ can be interpreted as the elements in graph equivariant cohomology.

LEMMA 4.1. *For $j = 1, \ldots, n+1$,*

$$x_j = M_{V_{2n}\setminus\{j+1\}} - M_{V_{2n}\setminus\{1\}} \in H^2(\mathcal{Q}_{2n}).$$

Because $H^{odd}(Q_{2n}) = 0$, as a module we have

$$H^*_{T^{n+1}}(Q_{2n}) \simeq H^*(Q_{2n}) \otimes_{\mathbb{Z}} H^*(BT^{n+1}).$$

Therefore, as a ring

$$H^*(Q_{2n}) \simeq H^*_{T^{n+1}}(Q_{2n})/\langle x_1, \ldots, x_{n+1}\rangle.$$

Consequently, together with Theorem 1.1 and Lemma 4.1, we obtain the following unified formula of two rings $H^*(Q_{4n})$ and $H^*(Q_{4n+2})$:

THEOREM 4.2 (ordinary cohomology). *There is the following isomorphism:*

$$H^*(Q_{2n}) \simeq \mathbb{Z}[\mathcal{Q}_{2n}]/\langle M_{V_{2n}\setminus\{j+1\}} - M_{V_{2n}\setminus\{1\}} \mid j = 1, \ldots, n+1\rangle.$$

4.2. $H^*(Q_{2n})$ from a combinatorial point of view. Using the relation $M_{V_{2n}\setminus\{j+1\}} - M_{V_{2n}\setminus\{1\}} = 0$ and the Relation 2, in $\mathbb{Z}[\mathcal{Q}_{2n}]/\langle M_{V\setminus\{j+1\}} - M_{V\setminus\{1\}} \mid j = 1, \ldots, n+1\rangle$, there is the following relation:

Relation 6. $M_I = M_{I'}$ for all $I, I' \subset V_{2n}$ with $|I| = |I'| = 2n+1$

Moreover, for $K \subset V_{2n}$ such that $|K| = n+1$ and $\{i, \bar{i}\} \not\subset K$ for every $i \in V_{2n}$, i.e., $\Delta_K \in H^{2n}(\mathcal{Q}_{2n})$ can be defined, by using Relation 3, we have the following relations:

Relation 7. There are the following two relations:

(1) $\Delta_{K^c} = M_I^n - \Delta_K$ if $n \equiv 0 \mod 2$;
(2) $\Delta_{K^c} = \Delta_K$ if $n \equiv 1 \mod 2$.

Relation 7 shows the difference between $H^*(Q_{4n})$ and $H^*(Q_{4n+2})$. We shall explain these differences by $H^*(Q_6)$ and $H^*(Q_8)$.

4.2.1. *$H^*(Q_4)$ from a combinatorial point of view.* By using Relation 7 (2), in $H^*(Q_4)$, we know that the three subgraphs in Figure 13 define the same class in $H^4(Q_4)$.

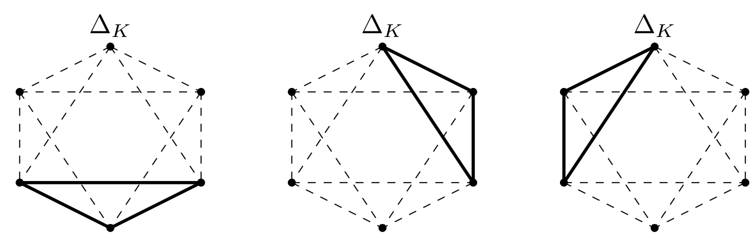

Figure 13. Three (same) classes in $H^4(Q_4)$.

Note that any pair of subgraphs in Figure 13 has always an intersection. This shows that $\Delta_K^2 (= x^2) \neq 0$ in $H^*(Q_4) \simeq \mathbb{Z}[c,x]/\langle c^3 - 2cx, x^2 - c^2x\rangle$.

On the other hand, $M_I^2 - \Delta_K$ can be illustrated as in Figure 14.

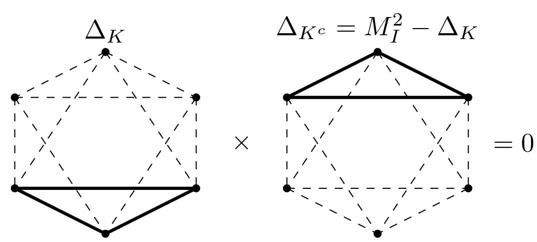

Figure 14. $\Delta_K(M_I - \Delta_K) = 0$.

Note that the subgraphs in Figure 14 have no intersections. Therefore, by using Relation 1, there exists the relation $\Delta_K(M_I^2 - \Delta_K)(= x^2 - c^2x) = 0$ in $H^*(Q_4) \simeq \mathbb{Z}[c,x]/\langle c^3 - 2cx, x^2 - c^2x\rangle$.

4.2.2. *$H^*(Q_6)$ from a combinatorial point of view.* By using Relation 7 (1), in $H^*(Q_6)$, for example, the two subgraphs in Figure 15 have an intersection, i.e., the multiplication of these classes are non-zero.

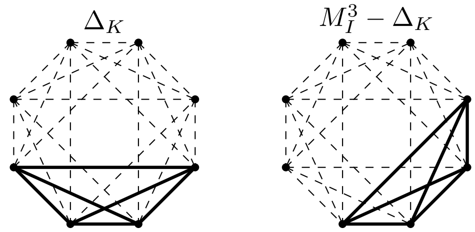

FIGURE 15. Two classes Δ_K and $M_I^3 - \Delta_K$ in $H^6(Q_6)$.

Note that any pair of subgraphs obtained by Δ_K and $M_I^3 - \Delta_K$ (see Figure 15) has always an intersection. This shows that $\Delta_K(M_I^3 - \Delta_K)(= x^2 - c^3x) \neq 0$ in $H^*(Q_6) \simeq \mathbb{Z}[c,x]/\langle c^4 - 2cx, x^2\rangle$.

On the other hand, Δ_K can be also obtained by the subgraph as in Figure 16.

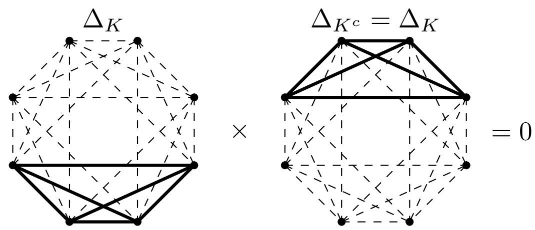

FIGURE 16. Figure shows the realtion $\Delta_K^2 = x^2 = 0$ in $H^*(Q_6)$.

Figure 16 shows that the class Δ_K is also identified with the class Δ_{K^c} in $H^*(Q_6)$. Therefore, by Relation 1, there is the relation $\Delta_K^2(= x^2) = 0$ in $H^*(Q_6) \simeq \mathbb{Z}[c,x]/\langle c^4 - 2cx, x^2\rangle$.

5. The problem inspired by algebraic geometry

We end this paper by asking about the related problem of the main theorem in this paper.

PROBLEM 5.1. *Let (Γ, α) be a GKM graph. Can every element $x \in H^*(\Gamma, \alpha)$ be written by the linear combinations of classes defined by generalized GKM subgraphs?*

Here, a class defined by a generalized GKM subgraph[5] seems to be a Thom class of the ordinary GKM subgraphs. This problem reminds us of the following question (this sentence is quoted from [**EH13**, Appendix C.2.4 "The Hodge conjecture"]):

- "the question of which cohomology classes on a smooth projective variety X can be represented as linear combinations of the fundamental classes of algebraic varieties; that is, what is the image of $\eta : A(X) \to H^*(X)$?"

For a GKM graph (Γ, α), the counterpart of the Chow ring $A(X)$ is a ring defined by some GKM subgraphs in GKM graph (Γ, α), and the counterpart of the cohomology ring $H^*(X)$ is the graph equivariant cohomology $H^*(\Gamma, \alpha)$.

Problem 5.1 is affirmatively solved for the case when (Γ, α) is a torus graph by Maeda-Masuda-Panov [**MMP07**] or more general orbifold torus graph by Darby-Kuroki-Song [**DKS22**]. They introduce the face ring $\mathbb{Z}[\Gamma, \alpha]$ of a(n) (orbifold) torus graph which is defined by all (orbifold) GKM subgraphs in a(n) (orbifold) torus graph, and they prove that $\mathbb{Z}[\Gamma, \alpha] \simeq H^*(\Gamma, \alpha)$. This result shows that all elements in $H^*(\Gamma, \alpha)$ can be represented as the linear combinations of Thom classes of (orbifold) GKM subgraphs.

The main theorem of the present paper also answers to Problem 5.1 affirmatively for the case when $(\Gamma, \alpha) = \mathcal{Q}_{2n}$ by introducing a ring $\mathbb{Z}[\mathcal{Q}_{2n}]$ in Definition 3.11 which is generated by different

[5]The definition is still vague but we do not want to use the global classes defined by the element $x \in H^*(\Gamma, \alpha)$ such that $x(p) \neq 0$ for all vertices $p \in V(\Gamma)$. For example, the Chern classes of tautological line bundle defined in [**KS**] are such global classes. Also, see the generator x in Figure 18 in Appendix A.

types of generators and relations from Maeda-Masuda-Panov's and Darby-Kuroki-Song's. Note that the generators in this paper are defined from the subgraphs in $\mathcal{Q}_{2n}$. Moreover, they are genuine GKM subgraphs, called Δ_K, or non-GKM subgraphs in the usual sense, called M_I (see Section 3). Geometrically, Δ_K is nothing but the (equivariant) Thom class of some smooth subvariety (see Remark 3.8), and M_I corresponds to some non-smooth subvarieties (isomorphic to the Schubert varieties (also see [**La72**])). The definition of M_I is purely combinatorics but this class is uniquely determined (as the "minimal" class which is only non-zero on the vertices $I = V_{2n} \setminus \{i\}$). So there should be a nice geometric (or cohomological) interpretation.

Appendix A. GKM description for non-effective T^1-actions on $\mathbb{C}P^1$

Let $T^1(= T)$ be the 1-dimensional torus. For every T^1-action on $\mathbb{C}P^1$, there exists a non-negative integer n such that the action is weak (i.e., up to the automorphism on T^1) equivariantly diffeomorphic to the following action:

$$t \cdot [z_0 : z_1] = [z_0 : t^n z_1],$$

where $t \in T^1$ and $[z_0 : z_1] \in \mathbb{C}P^1$. We denote this action as φ_n and the equivariant cohomology $H^*_T(\mathbb{C}P^1)$ with respect to this action as $H^*_{\varphi_n}(\mathbb{C}P^1)$. In [**KKLS20**, Remark 4.5], we show that

$$H^*_{\varphi_1}(\mathbb{C}P^1) \simeq \mathbb{Z}[\tau_1, \tau_2]/\langle \tau_1\tau_2 \rangle \not\simeq H^*_{\varphi_2}(\mathbb{C}P^1) \simeq \mathbb{Z}[u, v]/\langle u^2 - v^2 \rangle.$$

In this Appendix A, we show the GKM description of $H^*_{\varphi_n}(\mathbb{C}P^1)$ for all $n \geq 1$.

The Mayer-Vietoris exact sequence of the equivariant cohomology satisfies that

$$\cdots \longrightarrow H^j_{\varphi_n}(\mathbb{C}P^1) \longrightarrow H^j_{\varphi_n}(U_0) \oplus H^j_{\varphi_n}(U_1) \longrightarrow H^j_{\varphi_n}(U_0 \cap U_1) \longrightarrow H^{j+1}_{\varphi_n}(\mathbb{C}P^1) \longrightarrow \cdots$$

where $U_0 \simeq \{[z_0 : 1] \mid z_0 \in \mathbb{C}\}$ is the invariant open neighborhood of the fixed points $[0 : 1]$, $U_1 \simeq \{[1 : z_1] \mid z_1 \in \mathbb{C}\}$ is that of the fixed points $[1 : 0]$, and $U_0 \cap U_1 \simeq \{[z_0 : z_1] \mid z_0 z_1 \neq 0\} \simeq \mathbb{C}^*$. Since U_i is equivariantly contractible to the point and $U_0 \cap U_1$ is equivariant deformation retract to the great circle S^1, this sequence is isomorphic to the following sequence:

$$0 \longrightarrow H^{2j-1}_T(S^1) \longrightarrow H^{2j}_T(\mathbb{C}P^1) \longrightarrow H^{2j}(BT) \oplus H^{2j}(BT) \longrightarrow H^{2j}_T(S^1) \longrightarrow 0$$

Note that $H^*_T(S^1)$ is the equivariant cohomology of the n times rotated action of T^1 on S^1. Therefore, the T^1-action φ_n on S^1 has the kernel $\mathbb{Z}_n$ for $n \geq 2$, $\{e\}$ for $n = 1$. By the spectral sequence argument, we have that for $n \geq 2$

$$H^*_T(S^1) = H^*(ET \times_T S^1) \simeq H^*(ET/\mathbb{Z}_n) \simeq H^*(B\mathbb{Z}_n) \simeq \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}_n & * = 2j, j > 0 \\ 0 & * = 2j - 1 \end{cases}$$

Because $H^*(BT) \simeq \mathbb{Z}[x]$, we have the following short exact sequence

$$0 \longrightarrow H^{2j}_{\varphi_n}(\mathbb{C}P^1) \to \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}_n \longrightarrow 0$$

for $j > 0$ and $n \geq 2$. Hence, by the definition of the Mayer-Vietoris exact sequence, for all $n \geq 2$

$$\begin{aligned} H^*_{\varphi_n}(\mathbb{C}P^1) &\simeq \{f \oplus g \in \mathbb{Z}[x] \oplus \mathbb{Z}[x] \mid f_0 = g_0, f_j - g_j \equiv 0 \mod n\} \\ &\simeq \{f \oplus g \in \mathbb{Z}[x] \oplus \mathbb{Z}[x] \mid f - g \equiv 0 \mod nx\}. \end{aligned}$$

Note that for $n = 1$ this is nothing but the GKM description in the usual sense, i.e.,

$$H^*_{\varphi_1}(\mathbb{C}P^1) \simeq \{f \oplus g \in \mathbb{Z}[x] \oplus \mathbb{Z}[x] \mid f - g \equiv 0 \mod x\}.$$

Figure 17 shows the labeled graph which corresponds to φ_n. Note that φ_0 represents the trivial T-action on $\mathbb{C}P^1$.

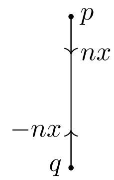

Figure 17. The GKM graph of φ_n, where $p = [1 : 0]$ and $q = [0 : 1]$. The element $x \in \mathfrak{t}^* \simeq \mathbb{R}$ is a generator of $\mathfrak{t}^*_{\mathbb{Z}} \simeq \mathbb{Z}$.

In summary, we have the following GKM description for φ_n and its ring structure.

Theorem A.1 (GKM description for non-effective torus action on $\mathbb{C}P^1$). *For every non-trivial T^1-action on $\mathbb{C}P^1$, there is the following ring isomorphism:*

$$H^*_{\varphi_n}(\mathbb{C}P^1) \simeq \{h : \{p, q\} \to \mathbb{Z}[x] \mid h(p) - h(q) \equiv 0 \mod nx\},$$

where $\{p, q\}$ is the fixed points for $n \geq 1$.

Furthermore, there is the following ring isomorphism:

$$H^*_{\varphi_n}(\mathbb{C}P^1) \simeq \mathbb{Z}[\tau_p, \tau_q, x]/\langle \tau_p\tau_q, nx - \tau_p + \tau_q\rangle$$

for $n \geq 0$, where τ_p, τ_q are the equivariant Thom classes of fixed points.

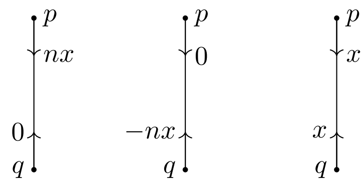

Figure 18. Figure of generators τ_p, τ_q, x (from left).

Acknowledgment

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山形大学名誉教授の内田伏一先生が 2021 年 12 月 9 日にお亡くなりになりになりました。内田伏一先生は私が山形大学に在籍時代の学部・修士時代（2000 年～2002 年）にかけての指導教官です。私は先生が退官される直前の学生でした。先生の元で直接勉強した学生の中では私がアカデミックの世界に残っている最後の一人になります。この場をお借りして日本の変換群論の発展に貢献されてきた先生の業績や個人的な思い出を振り返りたいと思います。

内田先生は東北大を卒業後、大阪大学に赴任されました。川久保勝夫先生達とともに日本の変換群論研究を牽引して来られた先生の一人です。研究者としての初期のころは、空間のはめ込みの研究をされていたようです。大阪大学のときには、コボルディズム論やコンパクトリー群の作用に関する研究をされていました。山形大学に赴任されてからは非コンパクトリー群の可微分な作用に関する論文を執筆され、twisted linear action と呼ばれる球面上への非コンパクトリー群の可微分な作用を組織的に構成する方法を定義されていました。山形大学において退官されるまで、それに関する論文を長年にわたり執筆されていました。退官後は魔方陣に関する研究を精力的にされていたようです。魔方陣に関する本も執筆されていました。著書も沢山あり、紀伊國屋数学叢書の『変換群とコボルディズム論』の他、裳華房から出ている『集合と位相』は今でも多くの大学の学部の教科書として定評があります。他にも多くの教科書を執筆されていました。このような業績から内田先生は変換群論のみならず日本のトポロジーの発展や大学数学の教育にも多くの貢献したと言って

も過言ではないと思います。私が学生の頃のトポロジーシンポジウムで当時九州大学にいた加藤十吉先生が「少し上の年代の内田先生にあこがれてトポロジーの研究を始めた」と話しておられたのを覚えています。また、先生がお亡くなりになったことをお聞きしてご自宅にお伺いしました。その折に、奥様から、「学生の話をするのがとても好きな人だった」ともお聞きしました。実際、私が内田先生を指導教官として選んだ理由の一つは講義がとても明快だったということです。

今回の研究の動機となった complex quadrics は内田先生のところにいたときに初めて勉強したものです。私が修士に上がる直前に渡された論文 [**Uc77**] は「初期の傑作のひとつ」と言って別刷りを渡されました。実際、MathSciNet で調べるとこの論文が内田先生の論文の中で最も引用されていることがわかります。Complex quadrics はこの論文の中で重要な役割を果たします。私が修士課程の頃はこの論文を何度も読み返しました。当時の私の知らないいろんな数学が使われていて、論文一つから様々なことを勉強できました。私の（大阪市大での）博士論文の一部は、complex quadrics に関する結果になりました。今回の研究はそのころからいつかやってみたいと思っていた研究でした。

最後に、私が将来数学者になりたいと内田先生に言った後に、質問に行くたびに「こんなのもわからないのか。困ったなー。」と頭を抱えられていたのですが、一度だけ「数学者になるために必要なのは才能よりも運だ。運というのはこれから君がどんな人と出会っていくかということだ。」と言われたのは一番印象に残っています。内田先生との出会いは、私にとっては本当に幸運な出会いの一つでした。学部・修士の頃のゼミでは毎回厳しいながらも、数学以外にもいろんなことを教えていただくことができました。この場をお借りして心から感謝を申し上げたいと思います。

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