On topology of toric spaces arising from 2-truncated cubes

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Toric varieties

Theorem (M.Atiyah; V.Guillemin, S.Sternberg'82)

Let (M, ω) be a $2d$-dimensional compact connected symplectic manifold with a hamiltonian action of a compact torus $\mathbb{T}^n$. Then the image of the moment map $\mu : M \to \mathbb{R}^n$ is a convex polytope P which is the convex hull of $\mu(M^{\mathbb{T}})$.

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If $d = n$ and the torus action is effective, then (M, ω) is a symplectic toric manifold.
A polytope P in $\mathbb{R}^n$ is called Delzant if its normal fan is smooth.

Theorem (T.Delzant'88)

There is a 1-1 correspondence between compact symplectic toric manifolds (M, ω, μ) (up to equivariant symplectomorphism) and Delzant polytopes $\mu(M)$ (up to lattice isomorphism).

Quasitoric manifolds

Definition

Let P be a combinatorial simple polytope of dimension n. A *quasitoric manifold* over P is a smooth $2n$-dimensional manifold M with a smooth action of the torus T^n satisfying the two conditions:
(1) the action is locally standard;
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Remarks

(a) M/T is homeomorphic, as a manifold with corners, to the simple polytope P;
(b) The action is free over the interior of P, the vertices of P correspond to the fixed points of the torus action on M;
(c) A projective toric manifold is a quasitoric manifold.

A moment-angle manifold $\mathcal{Z}_P$

In the work of M.Davis and T.Januszkiewicz'91 the following construction appeared.

Definition

Suppose P^n is a combinatorial simple polytope with facets $F_1, \ldots, F_m$. Denote by T^{F_i} a 1-dimensional coordinate subgroup in $T^F \cong T^m$ for each $1 \leq i \leq m$ and $T^G = \prod T^{F_i} \subset T^F$ for a face $G = \cap F_i$ of a polytope P^n. Then the *moment-angle manifold* corresponding to P is a quotient space

$$\mathcal{Z}_P = T^F \times P^n / \sim,$$

where $(t_1, p) \sim (t_2, q)$ iff $p = q \in P$ and $t_1 t_2^{-1} \in T^{G(p)}$, $G(p)$ is a minimal face of P which contains $p = q$.

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Simple polytopes

Now consider *simple convex n-dimensional polytopes* P in the Euclidean space $\mathbb{R}^n$ with scalar product $\langle\ ,\ \rangle$.

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Now consider *simple convex n-dimensional polytopes P* in the Euclidean space $\mathbb{R}^n$ with scalar product $\langle\ ,\ \rangle$.
Such a polytope P can be defined as a bounded intersection of m halfspaces:

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i \geq 0 \quad \text{for } i = 1, \ldots, m\}, \qquad (*)$$

where $\boldsymbol{a}_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$. We assume that the hyperplanes defined by the equations $\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i = 0$ are in general position, that is, at most n of them meet at a single point. We also assume that there are no redundant inequalities in $(*)$, that is, no inequality can be removed from $(*)$ without changing P.

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Then P has exactly m facets given by

$$F_i = \{\boldsymbol{x} \in P\colon \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i = 0\}, \quad \text{for } i = 1, \ldots, m.$$

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Let A_P be the $m \times n$ matrix of row vectors $\boldsymbol{a}_i$, and let $\boldsymbol{b}_P$ be the column vector of scalars $b_i \in \mathbb{R}$. Then we can write $(*)$ as

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon A_P\boldsymbol{x} + \boldsymbol{b}_P \geq \boldsymbol{0}\},$$

and consider the affine map

$$i_P \colon \mathbb{R}^n \to \mathbb{R}^m, \quad i_P(\boldsymbol{x}) = A_P\boldsymbol{x} + \boldsymbol{b}_P.$$

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It embeds P into

$$\mathbb{R}^m_{\geq} = \{\boldsymbol{y} \in \mathbb{R}^m \colon y_i \geq 0 \quad \text{for } i = 1, \ldots, m\}.$$

A moment-angle manifold $\mathcal{Z}_P$

Definition: V.Buchstaber and T.Panov (1998)

We define the space $\mathcal{Z}_P$ from the commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m \\ \downarrow & & \downarrow \mu \\ P & \xrightarrow{i_P} & \mathbb{R}^m_{\geq} \end{array}$$

where $\mu(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2)$. The latter map may be thought of as the quotient map for the coordinatewise action of the standard torus

$$\mathbb{T}^m = \{z \in \mathbb{C}^m \colon |z_i| = 1 \quad \text{for } i = 1, \ldots, m\}$$

on $\mathbb{C}^m$. Therefore, $\mathbb{T}^m$ acts on $\mathcal{Z}_P$ with quotient P, and i_Z is a $\mathbb{T}^m$-equivariant embedding.

Remarks

- If P_1 and P_2 are *combinatorially equivalent*, i.e. their face lattices are isomorphic, then $\mathcal{Z}_{P_1}$ and $\mathcal{Z}_{P_2}$ are homeomorphic. The opposite statement is **not** true (truncation polytopes);

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- If P_1 and P_2 are *combinatorially equivalent*, i.e. their face lattices are isomorphic, then $\mathcal{Z}_{P_1}$ and $\mathcal{Z}_{P_2}$ are homeomorphic. The opposite statement is **not** true (truncation polytopes);
- For any quasitoric manifold $M^{2n} \to P$ over a simple polytope P there is a principal T^{m-n}-bundle $\mathcal{Z}_P \to M^{2n}$, s.t. the composition $\mathcal{Z}_P \to M^{2n} \to P$ is a projection onto the orbit space of the T^m-action on $\mathcal{Z}_P$.

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Examples

- 1) If $P = \Delta^n$ then $\mathcal{Z}_P = S^{2n+1}$;
  2) If $P = P_1 \times P_2$ then $\mathcal{Z}_P = \mathcal{Z}_{P_1} \times \mathcal{Z}_{P_2}$

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- Consider a prism $Pr_3 = vc^1(\Delta^3)$, $\mathcal{Z}_{Pr_3} = S^3 \times S^5$ and cut a vertical edge. We get a 3-cube C, for which $\mathcal{Z}_C = S^3 \times S^3 \times S^3$.

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  If we perform an edge truncation of C we get a 5-gonal prism Pr_5 and $\mathcal{Z}_{Pr_5} = (S^3 \times S^4)^{\#5} \times S^3$.

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  If we perform an edge truncation of C we get a 5-gonal prism Pr_5 and $\mathcal{Z}_{Pr_5} = (S^3 \times S^4)^{\#5} \times S^3$.
- Consider a 3-polytope $P = vc^1(C)$. Then $\mathcal{Z}_P$ is **not** homotopy equivalent to a connected sum of products of spheres.

Motivation: formality

The moment-angle functor $\mathcal{Z}$ represents the homotopy type of $\mathcal{Z}_P$ and the ring structure of $H^*(\mathcal{Z}_P;\mathbb{k})$ as invariants of the combinatorial type (face lattice equivalence) of P.

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The moment-angle functor $\mathcal{Z}$ represents the homotopy type of $\mathcal{Z}_P$ and the ring structure of $H^*(\mathcal{Z}_P; \mathbb{k})$ as invariants of the combinatorial type (face lattice equivalence) of P.
Here we are mainly interested in the following problem:

Formality and higher Massey products for $\mathcal{Z}_P$

Determine the widest possible class of simple polytopes P s.t. there are nontrivial higher Massey operations in $H^*(\mathcal{Z}_P; \mathbb{Q})$, or more generally, $\mathcal{Z}_P$ is not *rationally formal*. Formality means, that its Sullivan-de Rham algebra (A, d) of PL-forms with coefficients in $\mathbb{Q}$ is formal in CDGA, i.e., there exists a zigzag of quasi-isomorphisms (weak equivalence) between (A, d) and its cohomology algebra $(H^*(\mathcal{Z}_P; \mathbb{Q}), 0)$.

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- quasitoric manifolds (T.Panov, N.Ray'08).

Moreover, formality is preserved by wedges, direct products and connected sums.

Stanley-Reisner rings

Let $\mathbb{k}$ be a commutative ring with a unit and consider a $(n-1)$-dimensional simplicial complex K on the ordered set $[m] = \{1, \ldots, m\}$. Let $\mathbb{k}[m] = \mathbb{k}[v_1, \ldots, v_m]$ be the graded polynomial algebra on m variables, $\deg(v_i) = 2$.

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Face rings

A *face ring* (or a *Stanley-Reisner ring*) of K is the quotient ring

$$\mathbb{k}[K] := \mathbb{k}[v_1, \dots, v_m]/\mathcal{I}_K$$

where $\mathcal{I}_K$ is the ideal generated by those square free monomials $v_{i_1} \cdots v_{i_s}$ for which $\{i_1, \dots, i_s\}$ is **not** a simplex of K. We denote $\mathbb{k}[P] = \mathbb{k}[\partial P^*]$.

Note that $\mathbb{k}[K]$ is a module over $\mathbb{k}[v_1, \dots, v_m]$ via the quotient projection.

Cohomology ring of $\mathcal{Z}_P$

The following result relates cohomology of $\mathcal{Z}_P$ to combinatorics of the polytope P:

Theorem (V.Buchstaber, T.Panov'98)

If we define a differential graded algebra
$R(P) = \Lambda[u_1, \ldots, u_m] \otimes \mathbb{k}[P]/(v_i^2 = u_i v_i = 0, 1 \leq i \leq m)$ with $\mathrm{bideg}\, u_i = (-1, 2)$, $\mathrm{bideg}\, v_i = (0, 2)$; $du_i = v_i$, $dv_i = 0$, then:

$$H^{*,*}(\mathcal{Z}_P; \mathbb{k}) \cong H^{*,*}[R(P), d] \cong \mathrm{Tor}^{*,*}_{\mathbb{k}[v_1,\ldots,v_m]}(\mathbb{k}[P], \mathbb{k}).$$

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These algebras admit $\mathbb{N} \oplus \mathbb{Z}^m$-multigrading and we have

$$\mathrm{Tor}^{-i,2\mathbf{a}}_{\Bbbk[v_1,\ldots,v_m]}(\Bbbk[P], \Bbbk) \cong H^{-i,2\mathbf{a}}(R(P), d),$$

where $\mathrm{Tor}^{-i,2J}_{\Bbbk[v_1,\ldots,v_m]}(\Bbbk[P], \Bbbk) \cong \widetilde{H}^{|J|-i-1}(P_J; \Bbbk)$ for $J \subset [m]$. Here we denote $P_J = \cup_{j \in J} F_j$. The multigraded component $\mathrm{Tor}^{-i,2\mathbf{a}}_{\Bbbk[v_1,\ldots,v_m]}(\Bbbk[P], \Bbbk) = 0$, if $\mathbf{a}$ is not a $(0,1)$-vector of length m.

We now turn to a discussion of flag nestohedra.

Building set

Let $S = \{1, 2, \ldots, n+1\}$, $n \geq 2$. A *building set on S* is a family of subsets $B = \{B_k \subseteq S\}$, such that: 1) $\{i\} \in B$ for all $1 \leq i \leq n+1$; 2) if $B_i \cap B_j \neq \varnothing$, then $B_i \cup B_j \in B$.

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Nestohedra

Nestohedron is a simple convex n-dimensional polytope
$P_B = \sum_{B_k \in B} \Delta_{B_k}$, where $\Delta_{B_k} = \mathrm{conv}\{e_j | j \in B_k\} \subset \mathbb{R}^{n+1}$.

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Example: graph-associahedra

A graphical building set $B(\Gamma)$ for a graph Γ on the vertex set S consists of such B_k that Γ_{B_k} is a **connected subgraph** of Γ. Then $P_\Gamma = P_{B(\Gamma)}$ is called a *graph-associahedron*.

Graph-associahedra

Graph-associahedra were first introduced by M.Carr and S.Devadoss (2006) in their study of Coxeter complexes.

Examples

- Γ is a complete graph on $[n+1]$.
  Then $P_\Gamma = Pe^n$ is a **permutohedron**.
- Γ is a stellar graph on $[n+1]$.
  Then $P_\Gamma = St^n$ is a **stellahedron**.
- Γ is a cycle graph on $[n+1]$.
  Then $P_\Gamma = Cy^n$ is a **cyclohedron** (or Bott-Taubes polytope).
- Γ is a chain graph on $[n+1]$.
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Graph-associahedra are *flag* polytopes, i.e. if a number of facets has an **empty** intersection then some pair of these facets has an **empty** intersection. Moreover, they are Delzant polytopes (all nestohedra are, due to the result of A.Zelevinsky).

In order to work with combinatorial types of graph-associahedra we should describe the structure of their face lattices.

Face poset

Facets of P_Γ are in 1-1 correspondence with **non** maximal connected subgraphs of Γ.
Moreover, a set of facets corresponding to such subgraphs $\Gamma_{i_1}, \ldots, \Gamma_{i_s}$ has a **nonempty** intersection if and only if:

(1) For any two subgraphs $\Gamma_{i_k}, \Gamma_{i_l}$, either they do not have a common vertex or one is a subgraph of another;

(2) If any two of the subgraphs $\Gamma_{i_{k_1}}, \ldots, \Gamma_{i_{k_l}}, l \geqslant 2$ do not have common vertices, then their union graph is disconnected.

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- If $P = Pe^n$ then its facets $F_1 \cap F_2 \neq \varnothing$ if and only if the corresponding Γ_1 and Γ_2 are subgraphs of one another;
- If $\Gamma_i, 1 \leq i \leq r$ are connected components of Γ then $P_\Gamma = P_{\Gamma_1} \times \ldots \times P_{\Gamma_r}$.

Definition: special subgraphs in a graph

1) Suppose Γ is a graph. For any of its connected subgraphs γ one can compute the number $i(\gamma)$ of such connected subraphs $\tilde{\gamma}$ in Γ that either

$$\gamma \cap \tilde{\gamma} \neq \varnothing, \gamma, \tilde{\gamma}$$

or

$$\gamma \cap \tilde{\gamma} = \varnothing,$$

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and $\gamma \sqcup \tilde{\gamma}$ is a connected subgraph in Γ.
2) We denote by $i_{max} = i_{max}(\Gamma)$ the maximal value of $i(\gamma)$ over all connected subgraphs γ in Γ. A connected subgraph γ, on which i_{max} is achieved, will be called a **special subgraph**.

Graph-associahedra: bigraded Betti numbers

Using the face poset structure of P_Γ we get the following result:

Theorem

Let $P = P_\Gamma$ be a graph-associahedron of dimension $n \geq 3$ for a connected graph Γ. Then for $i > i_{max}$:

$$\beta^{-i,2(i+1)}(P) = 0.$$

Denote the number of special subgraphs in Γ by s. Then

$$\beta^{-i_{max},2(i_{max}+1)}(P) = s.$$

Bigraded Betti numbers: examples

For the 4 classical series of graph-associahedra the theorem gives the following values of i_{max} and s.

Associahedron

$$\beta^{-q,2(q+1)}(P) = \begin{cases} n+3, & \text{if } n \text{ is even;} \\ \frac{n+3}{2}, & \text{if } n \text{ is odd;} \end{cases}$$

$$\beta^{-i,2(i+1)}(P) = 0 \quad \text{for } i \geq q+1,$$

where $q = q(n)$ is:

$$q = q(n) = \begin{cases} \frac{n(n+2)}{4}, & \text{if } n \text{ is even;} \\ \frac{(n+1)^2}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

Bigraded Betti numbers: examples

Cyclohedron

$$\beta^{-q,2(q+1)}(P) = \begin{cases} 2n+2, & \text{if } n \text{ is even;} \\ n+1, & \text{if } n \text{ is odd;} \end{cases}$$

$$\beta^{-i,2(i+1)}(P) = 0 \quad \text{for } i \geq q+1,$$

where $q = q(n)$ is:

$$q = q(n) = \begin{cases} \frac{n(n+2)-2}{2}, & \text{if } n \text{ is even;} \\ \frac{(n+1)^2-2}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Bigraded Betti numbers: examples

Permutohedron

$$\beta^{-q,2(q+1)}(P) = \binom{n+1}{[\frac{n+1}{2}]}$$

$$\beta^{-i,2(i+1)}(P) = 0 \quad \text{for } i \geq q+1,$$

where $q = q(n) = 2^{n+1} - 2^{[\frac{n+1}{2}]} - 2^{[\frac{n+2}{2}]} + 1$

Stellahedron

$$\beta^{-q,2(q+1)}(P) = \binom{n}{[\frac{n}{2}]}$$

$$\beta^{-i,2(i+1)}(P) = 0 \quad \text{for } i \geq q+1,$$

where $q = q(n) = 2^{n} - 2^{[\frac{n}{2}]} - 2^{[\frac{n+1}{2}]} + [\frac{n+3}{2}]$.

Pontryagin algebra $H_*(\Omega\mathcal{Z}_P)$

Bigraded Betti numbers of the type $\beta^{-i,2(i+1)}(P)$ have another topological application by means of the following result.

Loop homology of moment-angle-manifolds

J.Grbić, T.Panov, S.Theriault, J.Wu (2012) proved that $\sum_{i=1}^{m-n}\beta^{-i,2(i+1)}(P)$ equals the minimal number of multiplicative generators of the Pontryagin algebra $H_*(\Omega\mathcal{Z}_P;\mathbb{k})$ for any flag simple polytope P.

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Remark: torsion in cohomology

Consider the principal $\mathbb{T}^{m-n}$-bundle $\mathcal{Z}_P \to M_P$ for $P^n = P_\Gamma$. For $P = Pe^n$ and $n \geq 5$ $H^*(\mathcal{Z}_P)$ may have an arbitrary finite group as a direct summand. On the other hand, due to Danilov-Jurkiewicz theorem, $H^*(M_P)$ is always free.

Massey k-products in $H^*[A, d]$

Defining system

Suppose (A, d) is a dga, $\alpha_i = [a_i] \in H^*[A, d]$ and $a_i \in A^{n_i}$ for $1 \leq i \leq k$. Then a *defining system* for $(\alpha_1, \ldots, \alpha_k)$ is a $(k+1) \times (k+1)$-matrix C, s.t. the following conditions hold:

(1) $c_{i,j} = 0$, if $i \geq j$,

(2) $c_{i,i+1} = a_i$,

(3) $a \cdot E_{1,k+1} = dC - \bar{C} \cdot C$ for some $a = a(C) \in A$, where $\bar{c}_{i,j} = (-1)^{deg\, c_{i,j}} \cdot c_{i,j}$.

This implies: $d(a) = 0$ and $a \in A^m$, $m = n_1 + \ldots + n_k - k + 2$.

Defining system

Suppose (A, d) is a dga, $\alpha_i = [a_i] \in H^*[A, d]$ and $a_i \in A^{n_i}$ for $1 \leq i \leq k$. Then a *defining system* for $(\alpha_1, \ldots, \alpha_k)$ is a $(k+1) \times (k+1)$-matrix C, s.t. the following conditions hold:

(1) $c_{i,j} = 0$, if $i \geq j$,

(2) $c_{i,i+1} = a_i$,

(3) $a \cdot E_{1,k+1} = dC - \bar{C} \cdot C$ for some $a = a(C) \in A$, where $\bar{c}_{i,j} = (-1)^{\deg c_{i,j}} \cdot c_{i,j}$.

This implies: $d(a) = 0$ and $a \in A^m$, $m = n_1 + \ldots + n_k - k + 2$.

Definition

A *Massey k-product* $\langle \alpha_1, \ldots, \alpha_k \rangle$ is said to be *defined*, if there exists a defining system C for it.
If so, this Massey product consists of all $\alpha = [a(C)]$ for each defining system C. It is called *trivial*, if $[a(C)] = 0$ for some C.

Massey k-products: examples

k=2

If $\langle \alpha_1, \alpha_2 \rangle$ is defined, then we have:

$$a = d(c_{1,3}) - \bar{a}_1 \cdot a_2.$$

k=3

If $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is defined, then we have:

$$a = d(c_{1,4}) - \bar{a}_1 \cdot c_{2,4} - \bar{c}_{1,3} \cdot a_3,$$

$$\begin{aligned} d(c_{1,3}) &= \bar{a}_1 \cdot a_2, \\ d(c_{2,4}) &= \bar{a}_2 \cdot a_3. \end{aligned}$$

k=4

If $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ is defined, then we have:

$$a = d(c_{1,5}) - \bar{a}_1 \cdot c_{2,5} - \bar{c}_{1,3} \cdot c_{3,5} - \bar{c}_{1,4} \cdot a_4,$$

$$\begin{aligned}
d(c_{1,3}) &= \bar{a}_1 \cdot a_2, \\
d(c_{1,4}) &= \bar{a}_1 \cdot c_{2,4} + \bar{c}_{1,3} \cdot a_3, \\
d(c_{2,4}) &= \bar{a}_2 \cdot a_3, \\
d(c_{2,5}) &= \bar{a}_2 \cdot c_{3,5} + \bar{c}_{2,4} \cdot a_4, \\
d(c_{3,5}) &= \bar{a}_3 \cdot a_4.
\end{aligned}$$

Remarks

(1) V.Buchstaber and V.Volodin (2011) constructed realizations of all flag nestohedra as **2-truncated cubes**, i.e. a result of a sequence of truncations of codimension 2 faces only, starting with a cube, and proved the Gal conjecture on γ-vectors for them;

Triple Massey products in $H^*(\mathcal{Z}_P)$

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(2) G.Denham and A.Suciu (2005) described 5 graphs, s.t. there is a nontrivial triple Massey product of 3-dimensional classes in $H^*(\mathcal{Z}_P)$ iff one of these graphs is an induced subgraph in $sk^1(\partial P^*)$. All such products are decomposable.

The following result holds for triple Massey products in the cohomology ring of $\mathcal{Z}_P$.

Theorem

Let P be a generalized associahedron of type A, $B(C)$, D, or $P = P_\Gamma$. Then there is a defined and nontrivial triple Massey product $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ of some classes $\alpha_i \in H^3(\mathcal{Z}_P)$, $i = 1, 2, 3$ if and only if P is a generalized associahedron or a graph-associahedron P_Γ and in the graph Γ there is a connected component on $n + 1 = 4$ vertices, different from the complete graph K_4. All such Massey products are decomposable.

Triple Massey products in $H^*(\mathcal{Z}_P)$

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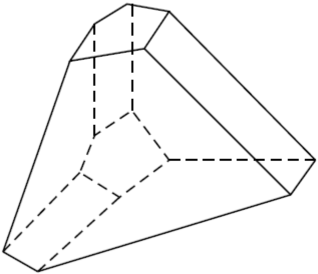
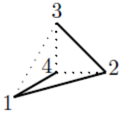
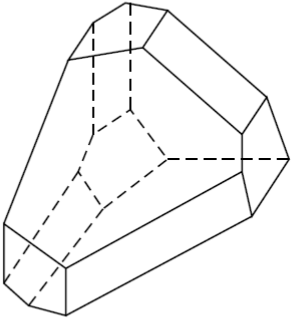
The face lattice allows us to reduce the general case to the case of $n = 3$ and apply the result of Denham and Suciu.

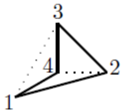
Example: generalized associahedron of type A or Stasheff polytope, $n = 3$





Example: generalized associahedron of type B (C) or Bott-Taubes polytope, $n = 3$



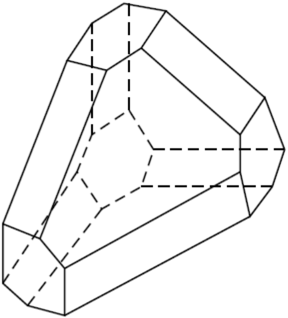


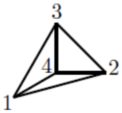
Any nestohedron on a connected building set can be obtained from a simplex as a result of a truncation sequence of the simplex's faces only.

Theorem

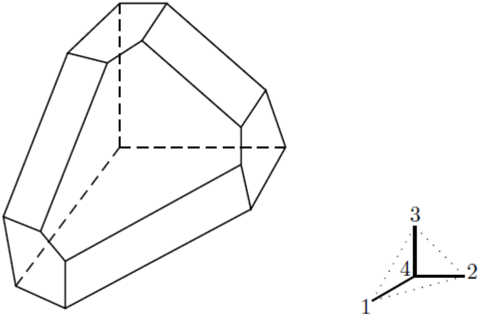
- If $P = Pe^n, n \geq 2$ and the classes $\alpha_i \in H^3(\mathcal{Z}_P), 1 \leq i \leq n+1$ are represented by $(n+1)$ pairs of the opposite permutohedra facets, then $\langle \alpha_1, \dots, \alpha_{n+1} \rangle$ is defined and trivial;
- If $P = St^n, n \geq 2$ and the classes $\alpha_i \in H^3(\mathcal{Z}_P), 1 \leq i \leq n$ are represented by n pairs of the opposite stellahedra facets, then $\langle \alpha_1, \dots, \alpha_n \rangle$ is defined and trivial.

Example: 3-dimensional permutohedron





Example: 3-dimensional stellahedron



Example: $n = 2$

- $P = As^2$ is a 5-gon, $\mathcal{Z}_P = (S^3 \times S^4)^{\#5}$ and the vanishing cup product corresponds to 2 pairs of non-adjacent edges in a 5-gon.
- $P = Pe^2$ is a 6-gon, $\mathcal{Z}_P = (S^3 \times S^5)^{\#6} \# (S^4 \times S^4)^{\#8} \# (S^5 \times S^3)^{\#3}$ and the vanishing triple Massey product corresponds to 3 pairs of parallel edges in a regular 6-gon.

We next consider a particular family of 2-truncated n-cubes $\mathcal{P}$, one for each dimension n, for which $\mathcal{Z}_{\mathcal{P}}$ has a nontrivial Massey product of order n.

Definition

Suppose I^n is an n-dimensional cube with facets $F_1, \ldots, F_{2n}$, such that F_i and F_{n+i}, $1 \leq i \leq n$ are parallel (do not intersect). Then we define $\mathcal{P}$ as a result of a consecutive cut of faces of codimension 2 from I^n, having the following Stanley-Reisner ideal:

$$I = (v_1 v_{n+1}, \ldots, v_n v_{2n}, v_1 v_{n+2}, \ldots, v_{n-1} v_{2n}, \ldots, v_1 v_{2n-1}, v_2 v_{2n}, \ldots),$$

or, equivalently,

$$I = (v_k v_{n+k+i}, 0 \leq i \leq n-2, 1 \leq k \leq n-i, \ldots),$$

where v_i correspond to F_i, $1 \leq i \leq 2n$ and in the last dots are the monomials corresponding to the new facets.

Remarks

- For $n = 2$ we get a 2-dimensional cube (the square) and for $n = 3$ we get a simple 3-polytope P with 8 facets giving a nontrivial triple Massey product due to I.Baskakov result (2003);

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- For $n = 2$ we get a 2-dimensional cube (the square) and for $n = 3$ we get a simple 3-polytope P with 8 facets giving a nontrivial triple Massey product due to I.Baskakov result (2003);
- $\mathcal{P}$ is a flag nestohedron: we can easily construct the building set B for $\mathcal{P}$ on the vertex set $S = [n+1]$ by identifying F_i with $\{1, \ldots, i\}$ for $1 \leq i \leq n$ and identifying F_i with $\{i - n + 1\}$ for $n + 1 \leq i \leq 2n$. Then we consecutively cut the following faces:

$$\{1\} \sqcup \{3\}, \{1, 2\} \sqcup \{4\}, \ldots, \{1, \ldots, n-1\} \sqcup \{n+1\}$$

$$\ldots$$

$$\{1\} \sqcup \{n\}, \{1, 2\} \sqcup \{n+1\}.$$

Remarks

- Thus, $\mathcal{P} = P_B$ for the building set B consisting of the building set
$$B_0 = \{\{i\}_1^{n+1}, \{1,2\}, \{1,2,3\}, \ldots, [n+1]\}$$
of an n-cube, the above subsets in $[n+1]$ and all the subsets in $[n+1]$ which are the unions of nontrivially intersecting elements in B;

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of an n-cube, the above subsets in $[n+1]$ and all the subsets in $[n+1]$ which are the unions of nontrivially intersecting elements in B;
- $\mathcal{P}$ is not a graph-associahedron: its number of facets $f_0(\mathcal{P}) = \frac{n(n+3)}{2} - 1 < f_0(As^n) = \frac{n(n+3)}{2}$, thus we can apply the lower and upper bounds for f-vectors of graph-associahedra proved by Buchstaber and Volodin.

Our main result on nontrivial higher Massey products for moment-angle manifolds is the following.

Theorem

Suppose $\alpha_i \in H^3(\mathcal{Z}_\mathcal{P})$ is represented by a 3-dimensional cocycle $v_i u_{n+i} \in R^{-1,4}(\mathcal{P})$ for $1 \leq i \leq n$ and $n \geq 2$. Then the n-fold Massey product $\langle\alpha_1, \dots, \alpha_n\rangle$ is defined and nontrivial.

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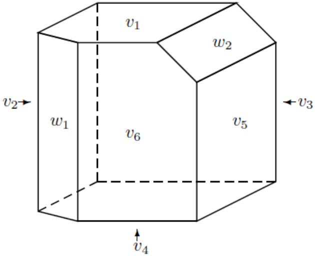
- Any element $\alpha = [a] \in \langle \alpha_1, \ldots, \alpha_n \rangle$ is s.t. $a \in R^*(\mathcal{P})$ is a sum of its multigraded components and $d : R^{-i,2J}(\mathcal{P}) \to R^{-(i-1),2J}(\mathcal{P})$. Thus, a is a coboundary iff each of its multigraded components is a coboundary;
- For any such $a \in R^*(\mathcal{P})$ its component in $R^{-(2n-2),2(1,\ldots,1,0,\ldots,0)}(\mathcal{P})$ with $2n$ "1"'s is always represented by the cocycle $v_1 v_{2n} u_2 u_3 \ldots u_{2n-1}$ (up to sign), which is not a coboundary.

Example: $n = 3$



$$I_P = (v_1 v_4, v_2 v_5, v_3 v_6, v_1 v_5, v_2 v_6, \ldots).$$

Then for $a_i = v_i u_{n+i}$ we have (up to sign):

$$c_{1,3} = v_1 u_2 u_4 u_5, c_{2,4} = v_2 u_3 u_5 u_6, a = v_1 v_6 u_2 u_3 u_4 u_5.$$

Thus, $\alpha = [a] = -[v_1 u_4 u_5] \cdot [v_6 u_2 u_3]$.

Using the previous theorem, the following statement can be obtained.

Theorem

There exists a flag nestohedron $P = P_B$, such that there are nontrivial higher Massey products of any prescribed orders $n_1, \ldots, n_r, r \geq 2$ in $H^*(\mathcal{Z}_P)$.

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Construction: substitution of building sets

Let $B_1, \ldots, B_{n+1}$ be connected building sets on $[k_1], \ldots, [k_{n+1}]$. Then, for every B on $[n+1]$, define $B' = B(B_1, \ldots, B_{n+1})$ on $[k_1] \sqcup \cdots \sqcup [k_{n+1}]$, consisting of elements $S^i \in B_i$ and $\bigsqcup_{i \in S} [k_i]$, where $S \in B$. Then $P_{B'} = P_B \times P_{B_1} \times \cdots \times P_{B_{n+1}}$.

We take $P = P_B$, $B = B'(B_1, \ldots, B_r)$, where $B_s, 1 \leq s \leq r$ is a building set for the corresponding $\mathcal{P}$ in the previous theorem and B' is a connected building set of a $(r-1)$-dimensional cube.

THANK YOU FOR YOUR ATTENTION!