FIRST PONTRJAGIN CLASSES OF MANIFOLDS HOMOTOPY EQUIVALENT TO $\mathbb{C}P(2k)$

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1. Introduction

The purpose of this note is to give a rough sketch of the proof of our main theorem.

Main Theorem

Let $f : M^{4k} \to \mathbb{C}P(2k)$ be a homotopy equivalence where M^{4k} is a closed smooth manifold. Then the difference of the fist Pontrjagin classes $\delta_1(M) = p_1(M) - f^*(p_1(\mathbb{C}P(2k)))$ is divisible by 16.

Notations used in this note:

- $\nu_p(n)$: the *p-order* of $n \in \mathbb{Z} \setminus \{0\}$ is the exponent of the prime p in the prime factorization of n. $\nu_p(0) := \infty$.
- $\nu_p(a/b) := \nu_p(a) - \nu_p(b)$ for $a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\}$.
- $\mathbb{Z}_{(p)}$: the ring of integers localized at the prime p, i.e. $\mathbb{Z}_{(p)} = \{a/b | a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\}, (b, p) = 1\}$.
- $\kappa_p(n) := \sum_i n_i$ for $n \in \mathbb{Z}_+, n = \sum_i n_i p^i$ and $0 \le n_i \le p-1$.
- $(f(x))^q = \sum_{i \ge 0} \alpha_i^{(q)} x^i$ for a formal power series $f(x) = \sum_{i \ge 0} \alpha_i x^i$.

We first present a short history of the partial but successful solutions to our problem.

- 1970, Brumfiel for $k = 2$ and 3, [1].
- 2003, K. for $k = 4$.
- 2004, Igarashi(Master thesis), $k \le 31$.
- 2009, K. for k with $\nu_2(k) \le 4$, published in 2012 [2].
- 2010, K. for k with $\nu_2(k) \le 5$, oral announcement.

The proof of our claim gets harder as the 2-order of k increases. If you go over a mountain, you face a much higher mountain standing in front of you. It seemed that there was no hope to go over all the infinite number of mountains in a finite lifetime. But in the fall of 2014, in a discussion of our joint work, we finally found a hint of trick to circumvent the obstacles to our final goal. Here is our story. We begin reviewing our research up to 2010.

(a) Let $f : M^{4h} \to \mathbb{C}P(2k)$ be a homotopy smoothing. Then there exists a fiber homotopically trivial vector bundle ζ over $\mathbb{C}P(2k)$ such that the tangent bundle $\tau(M)$ is stably isomorphic to the pull-back of $\tau(\mathbb{C}P(2k)) \oplus \zeta$ by f:
$$\tau(M) \overset{s}{\sim} f^*(\tau(\mathbb{C}P(2k)) \oplus \zeta).$$

(b) Using Hirzebruch's index theorem to M^{4k}, we see that
$$\mathrm{Index}(M) = \langle \mathcal{L}(M), [M] \rangle = \langle \mathcal{L}(\zeta)(x/\tanh x)^{2k+1}, [\mathbb{C}P(2k)] \rangle.$$
Since $\delta_1(M)$ is nothing but $f^*(p_1(\zeta))$, we have to study the bundle ζ when $\mathrm{Index}(M) = 1$.

(c) Let η be the canonical complex line bundle over $\mathbb{C}P(2k)$ and its first Chern class $c_1(\eta) = x$ is a generator of the cohomology ring $H^*(\mathbb{C}P(2k), \mathbb{Z}) = \mathbb{Z}[x]/(x^{2k+1})$. Let $\omega \in \widetilde{KO}(\mathbb{C}P(2k))$ be the realification of $\eta - 1_C \in \widetilde{K}(\mathbb{C}P(2k))$. It is known that $\widetilde{KO}(\mathbb{C}P(2k))$ is a free abelian group generated by ω^j $(j = 1, 2, \ldots, k)$. We take another set of generators $\psi_R^j(\omega)$ $(j = 1, 2 \ldots, k)$, where ψ_R^j is the real j-th Adams operation.

(d) According to the positive solution of the Adams-conjecture, the kernel of the J-map coincides with Image$(\psi_R^3 - 1)$ when localized at 2. Therefore when we put $\zeta_j = (\psi_R^3 - 1)\psi_R^j(\omega)$, then the fiber homotopically trivial vector bundle ζ can be written as

$$\zeta = m_1\zeta_1 + m_2\zeta_2 + \cdots + m_k\zeta_k,$$

where m_j belong to $\mathbb{Z}_{(2)}$. The Pontrjagin classes are calculated as follows:

$$p(\psi_R^j\omega) = 1 + j^2x^2,$$

$$p(\zeta_j) = \frac{1+(3j)^2x^2}{(1+j^2x^2)},$$

$$p(\zeta) = \prod_{j=1}^{k}\left(\frac{1+(3j)^2x^2}{1+j^2x^2}\right)^{m_j},$$

and

$$p_1(\zeta) = 8\sum_{j=1}^{k} j^2 m_j.$$

(e) To apply the index theorem, we introduce two power series $h(x)$ and $g(x)$:

$$h(x) = \frac{x}{\tanh x} = \sum_{i\geq 0} a_i x^{2i},$$

$$g(x) = \frac{1}{8}\left(\frac{h(3x)}{h(x)} - 1\right) = \sum_{i\geq 1} b_i x^{2i}.$$

$$\begin{aligned}
&\mathrm{Index}(M) = \langle \mathcal{L}(\zeta)h(x)^{2k+1}, [\mathbb{C}P(2k)]\rangle \\
&= \left(\mathcal{L}(\zeta)h(x)^{2k+1}\right)_{2k} = \left(\prod_{j=1}^{k}(1+8g(jx))^{m_j}h(x)^{2k+1}\right)_{2k} \\
&= 1 + 8\sum_{j=1}^{k} m_j (g(jx)h(x)^{2k+1})_{2k} \\
&\quad + \sum_{s\geq 2} 8^s \sum_{i_1+\cdots+i_k=s}\binom{m_1}{i_1}\cdots\binom{m_k}{i_k}\left(g(x)^{i_1}g(2x)^{i_2}\cdots g(kx)^{i_k}h(x)^{2k+1}\right)_{2k},
\end{aligned}$$

where $(f(x))_j$ denotes the coefficient of x^j in the formal power series $f(x)$. We shall use the following notations:

$$C(j_1, j_2, \cdots, j_s) = (g(j_1x)g(j_2x)\cdots g(j_sx)h(x)^{2k+1})_{2k},$$

$$D(i_1, i_2, \ldots, i_k) = C(\underbrace{1,\ldots,1}_{i_1}, \underbrace{2,\ldots,2}_{i_2}, \ldots, \underbrace{k,\ldots,k}_{i_k}).$$

Then we have

$$\mathrm{Index}(M) = 1 + 8\sum_{j=1}^{k} m_j C(j) + \sum_{s\geq 2} 8^s \sum_{i_1+\cdots+i_k=s}\binom{m_1}{i_1}\cdots\binom{m_k}{i_k}D(i_1,\ldots,i_k).$$

(f) Since Index$(M) = 1$, we have

$$\sum_{j=1}^{k} m_j C(j) + \sum_{s\geq 2} 8^{s-1} \sum_{i_1+\cdots+i_k=s}\binom{m_1}{i_1}\cdots\binom{m_k}{i_k}D(i_1,\ldots,i_k) = 0. \tag{1}$$

Out target is to show that $p_1(\zeta)$ is divisible by 16 from the condition (1). This is equivalent to the claim that $\sum_{j=1}^{k} j^2 m_j$ is even. This is also equivalent to $\sum_{j:odd} m_j$ is even. To simplify our notation we put $r = \nu_2(k)$. This proof can be obtained if one can prove the following three propositions:

Proposition A: If j is odd then $\nu_2(C(j)) = r$.

Proposition B: If j is even then $\nu_2(C(j)) \geq r + 1$.

Proposition C: If $s \geq 2$ then $\nu_2(C(j_1, \ldots, j_s) \geq r + 2 - 2s$.

And at the time of 2010, we were able to solve the problem under the condition that $r \leq 5$.

2. Outline of proof

We shall start attacking our Proposition A. We shall write $\nu_2(k) = r$. The following three lemmas were already known in 2010.

Lemma 1. *$C(1) = (3^k - (-1)^k)/(4 \cdot 3^k)$ and $\nu_2(C(1)) = r$.*

To prove the second part we used the fact $\nu_2(j^{2i} - 1) = \nu_2(j^2 - 1) + \nu_2(i)$ if j is odd.

Lemma 2.

$$\begin{aligned} C(\underbrace{1, \ldots, 1}_{s}) &= \left(\frac{1}{(3+x)^s(1-x)}\right)_{k-s} \\ &= \frac{1}{4^s}\left(\frac{1}{1-x} + \frac{1}{3+x} + \frac{4}{(3+x)^2} + \cdots + \frac{4^{s-1}}{(3+x)^s}\right)_{k-s} \\ &= \frac{1}{4^s 3^k}\left(3^k + (-1)^{k-s}\sum_{i=0}^{s-1}\binom{k-s+i}{i}3^{s-1-i}4^i\right). \end{aligned}$$

Lemma 3. (1) $\quad \nu_2(a_i) = \kappa_2(i) - 1$ *for all* $i \geq 0$,

(2) $\quad \nu_2(b_i) = \kappa_2(i) - 1$ *for all* $i \geq 1$.

Both results follow from the general Leibniz rule.

From here we present our tools discovered in 2014. All results are obtained by elementary methods.

Lemma 4. *Let $X_1, X_2, \ldots, X_t$ be variables and $0 \leq n \leq m$. Then*

$$(X_1 + X_2 + \cdots + X_t)^{p^m} \equiv (X_1^{p^{m-n}} + X_2^{p^{m-n}} + \cdots + X_t^{p^{m-n}})^{p^n} \mod p^{n+1}.$$

The proof of this lemma can be done by induction on n. Using this lemma, we can prove the following lemma that enables us to evaluate the 2-order of the coefficients of $h(x)^{2k}$.

Lemma 5. *Let $f(X) = \sum_{i \geq 0} \alpha_i X^i \in \mathbb{Z}_{(p)}[[X]]$. Then as to the coefficients of its q-th power $f(X)^q = \sum_i \alpha_i^{(q)} X^i$, we have $\nu_p(\alpha_i^{(q)}) \geq \nu_p(q) - \nu_p(i)$.*

Another tool is given by the next lemma.

Lemma 6. *Let $i_1, i_2, \ldots, i_s$ $(s \geq 2)$ be nonnegative integers and assume that $i_1 > 0$. Then*

$$\kappa_2(i_1) + \kappa_2(i_2) + \cdots + \kappa_2(i_s) \geq \nu_2(i_1 + i_2 + \cdots + i_s) + 1 - \nu_2(i_1).$$

The proof of the lemma for the special case $s = 2$ is done first. The general case follows from the special case and from the fact:

$$\kappa_2(i_2) + \cdots + \kappa_2(i_s) \geq \kappa_2(i_2 + \cdots + i_s).$$

Lemma 7. *If $s \geq 2$, then*

$$\nu_2(C(\underbrace{1, \ldots, 1}_{s})) \geq r + 2 - 2s.$$

From Lemma 2, we can show that

$$C(\underbrace{1, \ldots, 1}_{s}) = ((3^k - (-1)^k) + w(k))/(4^s 3^k),$$

where $w(k)$ is a polynomial in $4\mathbb{Z}_{(2)}[k]$ with $w(0) = 0$. Therefore $\nu_2(w(k)) \geq r + 2$. We also know that $\nu_2(3^k - (-1)^k) \geq r + 2$. This proves the lemma.

These last two lemmas are vitally important in proving our claims. Here we shall explain an outline of the proofs of our propositions A, B and C.

Proof of Proposition A. When $j = 1$, Proposition A is true by Lemma 1. We assume that j is odd and $j \neq 1$. Take the difference

$$\begin{aligned} C(j) - C(1) &= \left((g(jx) - g(x))\underline{h(x)^{2k+1}}\right)_{2k} \\ &= \left((g(jx) - g(x))\underline{h(x)h(x)^{2k}}\right)_{2k} \quad \text{Key trick!} \\ &= \left(\sum_{i_1 \geq 1}(j^{2i_1} - 1)b_{i_1}x^{2i_1}\sum_{i_2 \geq 0} a_{i_2}x^{2i_2}\sum_{i_3 \geq 0} a^{(2k)}_{i_3}x^{2i_3}\right)_{2k} \\ &= \sum_{i_1+i_2+i_3=k, i_1 \geq 1} (j^{2i_1} - 1)b_{i_1}a_{i_2}a^{(2k)}_{i_3}. \end{aligned}$$

Here we have

$$\begin{aligned} \nu_2((j^{2i_1} - 1)b_{i_1}a_{i_2}a^{(2k)}_{i_3}) &\geq (3 + \nu_2(i_1)) + (\kappa_2(i_1) - 1) + (\kappa_2(i_2) - 1) + \nu_2(a^{(2k)}_{i_3}) \\ &= \nu_{i_1} + \kappa_2(i_1) + \kappa_2(i_2) + 1 + \nu_2(a^{(2k)}_{i_3}) \\ &\geq \nu_2(k - i_3) + 2 + \nu_2(a^{(2k)}_{i_3}). \end{aligned}$$

We put $A = \nu_2(k - i_3) + 2 + \nu_2(a^{(2k)}_{i_3})$. If $i_3 = 0$, then we have $A = \nu_2(k) + 2 = r + 2$. If $0 < i_3$ and $\nu_2(i_3) \leq r$, then $\nu_2(k - i_3) \geq \nu_2(i_3)$ and $\nu_2(a^{(2k)}_{i_3}) \geq \nu_2(2k) - \nu_{i_3} = r + 1 - \nu_2(i_3)$. Therefore we have $A \geq r + 3$. If $\nu_2(i_3) > r$, then we have $\nu_2(k - i_3) = r$. Thus we have $A \geq r + 2$. This shows that $\nu_2(C(j) - C(1)) \geq r + 2$. Therefore we have $\nu_2(C(j)) = r$ when j is odd. This completes the proof of Proposition A.

Proof of Proposition B. When j is even, we have

$$C(j) = \sum_{i_1+i_2+i_3=k, i_1 \geq 1} j^{2i_1}b_{i_1}a_{i_2}a^{(2k)}_{i_3}.$$

We put $B = \nu_2(j^{2i_1}b_{i_1}a_{i_2}a^{(2k)}_{i_3})$. Then we have

$$B \geq 2i_1 + \kappa_2(i_1) + \kappa_2(i_2) - 2 + \nu_2(a^{(2k)}_{i_3}) = \nu_2(k - i_3) + 2i_1 - \nu_2(i_1) - 1 + \nu_2(a^{(2k)}_{i_3}).$$

From this it is not hard to see that $B \geq r + 1$. This proves Proposition B.

Proof of Proposition C. If there is at least one even number in $j_1, \ldots, j_s$, then the proof proceeds in the same manner as the proof of Proposition B. If all numbers $j_1, \ldots, j_s$ are odd, then we can show that

$$\nu_2(C(j_1, \cdots, j_s) - C(1, \ldots, 1)) \geq r + 2 - 2s$$

using a similar argument as in the proof of Proposition A.

References

[1] BRUMFIEL, G., Homotopy equivalences of almost smooth manifolds, Comment. Math. Helv. **46** (1971), 381–407.

[2] KITADA, Y., On the first Pontrjagin class of homotopy complex projective spaces, Math. Slovaca, 62(2012), No. 3, 551-566.

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