

Higher coherence and higher theories of algebraic structures

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A coloured operad or “multicategory” controls algebras over it, and is analogous in this respect, to a (multi-sorted) algebraic theory in the sense of William Lawvere. On the other hand, an algebra over a multicategory is a kind of structure definable in any multicategory so the notion becomes enriched.

Given a kind of algebraic structure in general, one may wonder what may control those structures, and where the notion may be enriched. For example, multicategories of a given kind such as symmetric, planar or braided, turn out to be controlled by a very simple “2-dimensional” algebraic structure, which we call “2-theory” inspired by Lawvere’s notion.

The notion of 2-theory is obtained from the notion of multicategory through a process which we call “theorization”. The notion of (symmetric) multicategory itself is a theorization of the notion of commutative algebra. A theorized form of a given kind of structure, or a “theory” as we shall call it, controls “algebras” over it, which generalize the untheorized structures as the “theory” varies. A theorized structure also generalizes a categorified structure as a place where the structures can be enriched. In fact, a general enriched notion of algebra will be (“coloured”) functor of theories.

Various kinds of structure can be theorized using inductivity embedded in the structure of the coherence for higher associativity. As a result of this inductivity, which leads to a complete description of the combinatorics of the higher coherence, the process of theorization can moreover be iterated endlessly to lead to the concept of higher theory.

We describe examples of theorized and related structures, and consequences of (iterative) theorization.