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Positively curved manifolds with isometric torus actions

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A question and a problem.

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What are the topological implications of positive sectional curvature?

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Problem

Classify manifolds admitting metrics of positive sectional curvature.

Examples of positively curved manifolds.

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Examples of positively curved manifolds.

- There are only very few examples of manifolds admitting metrics of positive sectional curvature.
- For $\dim M > 24$ all known examples are diffeomorphic to S^n, $\mathbb{C}P^n$, or $\mathbb{H}P^n$.
- Other examples are known in dimensions 6, 7, 12, 13 and 24
- These are certain homogeneous spaces and biquotient spaces.

Topological implications of positive curvature.

For closed manifolds M the following is known:

Classical results

- Theorem of Gauß-Bonnet: $\sec(M^2) > 0 \Rightarrow M$ is diffeomorphic to S^2 or $\mathbb{R}P^2$.

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- Theorem of Synge: $\sec(M^{2n}) > 0 \Rightarrow |\pi_1(M)| \leq 2$.
- Theorem of Bonnet-Myers: $\mathrm{Ric}(M^n) > 0 \Rightarrow |\pi_1(M)| < \infty$.

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- Theorem of Synge: $\sec(M^{2n}) > 0 \Rightarrow |\pi_1(M)| \leq 2$.
- Theorem of Bonnet-Myers: $\mathrm{Ric}(M^n) > 0 \Rightarrow |\pi_1(M)| < \infty$.
- Gromov's Betti number Theorem: $\sec(M^n) \geq 0 \Rightarrow \sum_i b_i(M) < C(n)$.

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- But there are the following conjectures:

Hopf's Conjecture I

If M is a closed, even-dimensional positively curved manifold, then the Euler characteristic of M is positive.

Hopf's Conjecture II

$S^2 \times S^2$ does not admit a positively curved metric.

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- The first conjecture is true in dimensions two and four.

A programme.

Grove's Programme

Classify simply connected positively curved manifolds with large isometry group first.

The result of Grove and Searle.

Theorem (Grove and Searle 1992)

Let M^n be positively curved and simply connected.
Assume that there is an isometric, effective action of a torus T^d with

$$d \geq \left[\frac{n+1}{2}\right].$$

Then M is diffeomorphic to S^n or $\mathbb{C}P^{\frac{n}{2}}$.

Wilking's Theorem.

Theorem (Wilking 2003)

Let M^n be manifold with $\pi_1(M) = 0$, $\sec(M) > 0$, and $n \geq 10$.
Suppose that there is an effective isometric action of a d-dimensional torus T^d on M^n with

$$d \geq \frac{1}{4}n + 1.$$

Then M^n is homeomorphic to $\mathbb{H}P^{\frac{n}{4}}$ or to S^n, or M is homotopy equivalent to $\mathbb{C}P^{\frac{n}{2}}$.

Main tool in the proof.

Connectedness Lemma (Wilking 2003)

Assume $\sec(M^n) > 0$. *If* $N^{n-k} \subset M^n$ *is a totally geodesic submanifold, then the inclusion* $N^{n-k} \to M^n$ *is* $(n-2k+1)$*-connected.*

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- Fang and Rong (2005) gave a homeomorphism classification of positively curved n-manifolds with symmetry rank $[\frac{n-1}{2}]$.
- Dessai (2007) gives vanishing results for coefficients of the elliptic genus of a positively curved two-connected manifold with isometric S^1-action.

Kennard's result.

Theorem (Kennard 2013)

Assume $\sec(M^n) > 0$. *If* $n \equiv 0 \mod 4$ *and M admits an effective, isometric* T^d*-action with*

$$d \geq 2\log_2 n - 2,$$

then $\chi(M) > 0$.

Main tool in proof.

Periodicity Theorem (Kennard 2012)

Let M^n with $\sec(M^n) > 0$, $\pi_1(M) = 0$.
Assume there is a pair of totally geodesic, transversely intersecting submanifolds of codimensions $k_1 \geq k_2$. If $k_1 + 3k_2 \leq n$, then $H^*(M;\mathbb{Q})$ is 4-periodic.

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Definition

Here $H^*(M)$ is called k-periodic, if there is a $e \in H^k(M)$ such that

$$\cup e \colon H^i(M) \to H^{i+k}(M)$$

is an isomorphism for all $0 \leq i \leq \dim M - k$ or $H^*(M) \cong H^*(S^{\dim M})$.

Remarks

- If $H^*(M;\mathbb{Q})$ is four-periodic and $b_1(M) = b_3(M) = 0$, then $H^*(M;\mathbb{Q})$ is one of the following:
  $H^*(S^n;\mathbb{Q})$, $H^*(\mathbb{C}P^{\frac{n}{2}};\mathbb{Q})$, $H^*(\mathbb{H}P^{\frac{n}{4}};\mathbb{Q})$, or $H^*(S^2 \times \mathbb{H}P^{\frac{n-2}{4}};\mathbb{Q})$.

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- If $H^*(M;\mathbb{Q})$ is four-periodic, $b_1(M) = 0$ and $n \equiv 0 \mod 4$, then $b_3(M) = 0$.

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- Weisskopf (2017) combined Kennard's result with the methods of Dessai.
- Goertsches and W. (2015) studied positively curved GKM-manifolds.

First main result.

Theorem (Kennard, W., Wilking 2019)

Assume $\sec(M^n) > 0$ *and that there is an isometric effective action of a torus* T^d *of dimension*

$$d \geq 5.$$

Then every component F *of* M^T *has the rational cohomology of* S^m, $\mathbb{C}P^m$ *or* $\mathbb{H}P^m$.

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- This is first result in this direction where the dimension of the acting torus does not grow with the dimension of the manifold.

Some corollaries.

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Corollary

The isometry group of a potential positively curved metric on $S^{2n+1} \times S^{2n'+1}$ has rank at most four.

New tool in proof.

T-Splitting Theorem

If $T^{2d+1} \to SO(V)$ *is a faithful representation,*

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T-Splitting Theorem

If $T^{2d+1} \to SO(V)$ is a faithful representation, then there exists a d-dimensional subgroup $H \subset T^{2d+1}$ such that the induced representation $T^{2d+1}/H \to SO(V^H)$ is faithful and has exactly $d+1$ non-trivial, pairwise inequivalent, irreducible subrepresentations.

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  - Let P be a component of M^H, where H is as in the T-Splitting Theorem (with $V = T_xM$ for $x \in F$).
  - Then there are four totally geodesic submanifolds of P intersecting pairwise transversely.
  - Applying the Periodicity Theorem to the two submanifolds of smallest codimension, implies that P has four-periodic cohomology.

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- Therefore by classical results one only has to deal with the case that $H^*(P;\mathbb{Q}) = H^*(S^2 \times \mathbb{H}P^n;\mathbb{Q})$.
- In this case computations in equivariant cohomology and the connectedness lemma lead to the desired result.

Second main result.

Theorem (Kennard, W., Wilking 2019)

Assume $\sec(M^n) > 0$, $\pi_1(M) = 0$. *If M admits an isometric effective equivariantly formal action of T^d, with*

$$d \geq 8,$$

then M has the rational cohomology of S^n, $\mathbb{C}P^{\frac{n}{2}}$, or $\mathbb{H}P^{\frac{n}{4}}$.

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then M has the rational cohomology of S^n, $\mathbb{C}P^{\frac{n}{2}}$, *or* $\mathbb{H}P^{\frac{n}{4}}$.

- Bott's Conjecture asks whether a positively curved manifold is rationally elliptic.
- Together with Hopf's Conjecture it would imply that $H^{\mathrm{odd}}(M;\mathbb{Q}) = 0$ in even-dimensions. This in turn implies equivariant formality.

A corollary.

Corollary

The isometry group of a potential positively curved metric on $S^{2n} \times S^{2n'}$ *and* $S^{2n} \times S^{2n'-1}$, $n' \leq n$ *has rank at most seven.*

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- For the proof note that all torus actions on these manifolds are equivariantly formal.
- There exist equivariantly non-formal actions on $S^2 \times S^3$.

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is injective.

- Hence it suffices to determine the image of ι^*.

Outline of proof II.

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Lemma (Chang, Skjelbred 1974)

Assume that the T-action on M is equivariantly formal. Then for every closed invariant subspace $M_1 \subset X \subset M$,

$$\iota^* H_T^*(X; \mathbb{Q}) = \iota^* H_T^*(M; \mathbb{Q}) \subset H_T^*(M^T; \mathbb{Q}).$$

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- Hence, it suffices to determine the combinatorial structure of M_1.

Outline of proof III.

In even dimensions n, we now have to consider two cases:

- $\exists T^7 \subset T^8$ and $F_0 \subset M^{T^7}$, $\dim F_0 \geq 4$.
- The T^8-action is GKM. (This case is similar to the discussion in Goertsches-W. 2015)

Outline of proof IV.

In the first case. Then:

- Using work of Smith, Bredon, Hsiang-Su,

$$M_1 \subset \bigcup_{T^5 \subset T^7} N_{T^5} =: X,$$

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- Using a Mayer-Vietoris argument, we show that

$$\iota^* H^*_{T^7}(X; \mathbb{Q}) \subset H^*(M^{T^7}; \mathbb{Q})$$

is isomorphic to a similar algebra constructed for a linear action on some S^n, $\mathbb{H}P^n$, or $\mathbb{C}P^n$, respectively.

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- The Chang–Skjelbred Lemma now implies the claim.

Thank you!