Foliations arising from configurations of vectors, Gale duality, and moment-angle manifolds

joint with Maxim Strumentov

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Vector configurations and their associated foliations

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This is a very classical dynamical system taking its origin in the works of Poincaré. There is a well-known relationship between the linear properties of Γ and the topology of the foliation of $\mathbb{R}^m$ by the orbits of the action. We attempt for systematising the existing knowledge on this relationship and proceed by analysing the topology of the quotient (the leaf space) using some recent constructions of toric topology.

Motivation

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- Topology of intersections of real and Hermitian quadrics (topology & holomorphic dynamics)
- The quotient construction of toric varieties (the Cox construction) (toric geometry)
- Smooth and complex-analytic structures on moment-angle manifolds (toric topology)

Example

Consider two actions of $V = \mathbb{R}$ on $\mathbb{R}^2$ given by

$$(v,(x_1,x_2)) \mapsto (e^{vx_1}, e^{vx_2}), \tag{1}$$

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For (2), the quotient $(\mathbb{R}^2 \setminus \{\mathbf{0}\})/\mathbb{R}$ is a non-Hausdorff space.

The difference is that (1) is a proper action, while (2) is not.

Nondegenerate leaves

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The orbit $V\mathbf{x}$ of a point $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$ under the action $V \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is free iff the subset $\{\gamma_i : x_i \neq 0\} \subseteq \Gamma$ spans the whole V^*.

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Proof.

Suppose the orbit $V\boldsymbol{x}$ is not free, i. e. there exists $\boldsymbol{v} \neq 0$ such that

$$(x_1 e^{\langle \gamma_1, \boldsymbol{v} \rangle}, \ldots, x_m e^{\langle \gamma_m, \boldsymbol{v} \rangle}) = (x_1, \ldots, x_m).$$

Then $\langle \gamma_i, \boldsymbol{v} \rangle = 0$ for $x_i \neq 0$, which implies that the vectors γ_i with $x_i \neq 0$ do not span V^*. The opposite statement is proved similarly. □

Denote $[m] = \{1, \ldots, m\}$ and consider subsets $I = \{i_1, \ldots, i_p\} \subseteq [m]$. For each I we denote

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If $\Gamma_{\widehat{I}}$ spans V^*, then so does $\Gamma_{\widehat{J}} \supset \Gamma_{\widehat{I}}$ for any $J \subset I$. Hence, $\mathcal{K}(\Gamma)$ is a simplicial complex. Also, if $\Gamma_{\widehat{I}}$ spans V^*, then it contains a basis of V^*. Such a basis has the form $\Gamma_{\widehat{L}}$ for some L with $I \subset L$ and $|L| = m - |\Gamma_{\widehat{L}}| = m - k$. It follows that each face $I \in \mathcal{K}$ is contained in a $(m - k - 1)$-dimensional face, so $\mathcal{K}(\Gamma)$ is pure $(m - k - 1)$-dimensional. □

Given a simplicial complex $\mathcal{K}$ on $[m]$, define the following open subset in $\mathbb{R}^m$ (the complement of an arrangement of coordinate subspaces):

$$U(\mathcal{K}) = \mathbb{R}^m \setminus \bigcup_{\{i_1,\ldots,i_p\}\notin\mathcal{K}} \{\boldsymbol{x} : x_{i_1} = \cdots = x_{i_p} = 0\}.$$

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For example, if $\mathcal{K} = \{\varnothing\}$, then $U(\mathcal{K}) = (\mathbb{R}^\times)^m$, where $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$, and if $\mathcal{K}$ consists of all proper subsets of $[m]$, then $U(\mathcal{K}) = \mathbb{R}^m \setminus \{\mathbf{0}\}$.

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We restate this by saying that $U(\mathcal{K})$ consists of nondegenerate leaves of the foliation defined by $V \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ for any $\mathcal{K} \subseteq \mathcal{K}(\Gamma)$.

Linear Gale duality

Given $\Gamma = (\gamma_1, \dots, \gamma_m)$, define a linear map $\mathit{\Gamma}\colon \mathbb{R}^m \to V^*$, $\boldsymbol{e}_i \mapsto \gamma_i$.
Let $W := \operatorname{Ker} \mathit{\Gamma}$, so we have dual exact sequences

$$0 \longrightarrow W \longrightarrow \mathbb{R}^m \xrightarrow{\mathit{\Gamma}} V^* \longrightarrow 0,$$

$$0 \longrightarrow V \xrightarrow{\mathit{\Gamma}^*} \mathbb{R}^m \xrightarrow{A} W^* \longrightarrow 0,$$

where $\mathit{\Gamma}^*$ takes $\boldsymbol{v}$ to $(\langle\gamma_1, \boldsymbol{v}\rangle, \dots, \langle\gamma_m, \boldsymbol{v}\rangle)$. Set $\boldsymbol{a}_i := A(\boldsymbol{e}_i)$.

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If we choose bases in V and W, then $\mathit{\Gamma}$ becomes a $k \times m$-matrix with columns $\gamma_1, \ldots, \gamma_m$ and A becomes an $(m-k) \times m$-matrix with columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$. The identity $A\mathit{\Gamma}^* = 0$ implies that the rows of A form a basis in the space of linear relations between the vectors $\gamma_1, \ldots, \gamma_m$.

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Let Σ be a simplicial fan in W^*, and let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ be generators of one-dimensional cones of Σ. The underlying simplicial complex $\mathcal{K} = \mathcal{K}_\Sigma$ is the collection of subsets $I \subseteq [m]$ such that $\{\boldsymbol{a}_i : i \in I\}$ spans a cone of Σ.

A simplicial fan Σ is therefore determined by two pieces of data:

- a simplicial complex $\mathcal{K}$ on $[m]$;
- a configuration of vectors $\mathrm{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ in W^* such that for any simplex $I \in \mathcal{K}$ the subset $\mathrm{A}_I = \{\boldsymbol{a}_i : i \in I\}$ is linearly independent.

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The 'bunch of cones' $\{\sigma_I : I \in \mathcal{K}\}$ patches into a fan Σ whenever any two cones σ_I and σ_J intersect in a common face (which has to be $\sigma_{I \cap J}$). Under this condition, we say that the data $\{\mathcal{K}, \mathrm{A}\}$ define a fan Σ.

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(c) $\operatorname{relint}\operatorname{cone}(\Gamma_{\widehat{I}}) \cap \operatorname{relint}\operatorname{cone}(\Gamma_{\widehat{J}}) \neq \varnothing$ *for any* $I, J \in \mathcal{K}$.

Proper actions

A continuous action $G \times X \to X$, $(g, x) \mapsto g\,x$ of a topological group G on a topological space X is proper if the map $h\colon G \times X \to X \times X$, $(g, x) \mapsto (g\,x, x)$ is proper, that is, $h^{-1}(C)$ is compact for any compact $C \subseteq X \times X$.

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- the quotient M/G of a proper action of a Lie group action G on a manifold M is Hausdorff;
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For our action $V \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ we have the following result:

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Let $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ be a vector configuration in V^* defining the action $V \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$, and let $\mathrm{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ be the Gale dual configuration. Let $\mathcal{K}$ be a simplicial complex on $[m]$ such that for any $I \in \mathcal{K}$ the subset $\Gamma_{\widehat{I}}$ spans V^* (equivalently, the subset A_I is linearly independent). Then

(1) *the restricted action $V \times U(\mathcal{K}) \to U(\mathcal{K})$ is free;*

(2) *the action $V \times U(\mathcal{K}) \to U(\mathcal{K})$ is proper iff $\{\mathcal{K}, \mathrm{A}\}$ define a fan.*

If $\{\mathcal{K}, \mathrm{A}\}$ define a complete fan in W^* (i. e. the union of all cones is the whole W^*), then the quotient $U(\mathcal{K})/V$ is a compact smooth manifold. It is known in toric topology as the real moment-angle manifold corresponding to $\mathcal{K}$.

Polytopal fans and intersections of quadrics

The normal fan Σ_P of a simple convex polytope P in W is an important example of a complete simplicial fan. In this case, the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ are the inward-pointing normals to the facets of P, and a subset A_I spans a cone iff the intersection of facets with normals $\boldsymbol{a}_i$, $i \in I$, is nonempty.

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(a) *Σ is a normal fan of polyhedron (respectively, polytope);*

(b) $\bigcap_{I\in\mathcal{K}} \mathrm{relint}\,\mathrm{cone}(\Gamma_{\widehat{I}}) \neq \varnothing$.

Therefore, the data $\{\mathcal{K}, \mathrm{A}\}$ define a fan Σ iff the relative interiors of Gale dual cones $\operatorname{cone} \Gamma_{\widehat{I}}$ have pairwise nonempty intersections, and Σ is the normal fan of a polytope iff all the cones $\operatorname{cone} \Gamma_{\widehat{I}}$ have a common relative interior point.

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Idea of proof.

The function $f\colon \mathbb{R}^m \to \mathbb{R}$, $f(\boldsymbol{x}) = \|\gamma_1 x_1^2 + \cdots + \gamma_m x_m^2 - \boldsymbol{c}\|^2$ has a unique minimum at each orbit $V\boldsymbol{x}$, and the set of these minima is the intersection of quadrics above. □

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Provided that the holomorphic action $V \times U(\mathcal{K}) \to U(\mathcal{K})$ is free and proper (the *fan condition*), the quotient $\mathcal{Z}_K = U(\mathcal{K})/V$ is a complex-analytic manifold (the complex moment-angle manifold).

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This construction leads to a new family on *non-Kähler* complex manifolds, which includes the classical series of Hopf and Calabi–Eckmann manifolds.

References (to some earlier works)

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