Topology of orbit spaces of complexity 1 torus actions

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Basic objects

Let us consider the following objects:

- M^{2n}: a connected compact smooth $2n$-manifold,
- $T^k = \underbrace{S^1 \times \ldots \times S^1}_{k}$: the compact k-torus,
- $\theta : T^k \curvearrowright M^{2n}$: an effective smooth action.

The number $d = n - k$ is called the complexity of the action of θ.

We also suppose the following conditions (*) to be satisfied:

- the set M^T of fixed points is finite and nonempty,
- for each point $x \in M^{2n}$, its stabilizer subgroup $St(x) \in T^k$ is connected,
- for each subgroup $H \subset T^k$, the closure of every connected component of the set $\{x \in M^{2n} | St(x) = H\}$ contains a point x' such that $\dim St(x`) > \dim H$.

Question: what can be said about the quotient M^{2n}/T^k?

Classical case $d = 0$

A classical idea for studying group actions is constructing an equivariant topological model for the original space. In our case:

$$M^{2n}_{Q,\lambda} = (Q \times T^n)/\sim$$

where $Q = M^{2n}/T^n$ and λ is the characteristic function, mapping an orbit $Tx \in Q$ to its stabilizer subgroup $St(x)$.

A classical object of study in toric topology is a class of manifolds called quasitoric manifolds. Characteristic data for recovering a quasitoric manifold $X = X^{2n}_{P,\lambda}$ consists of:

- a simple n-polytope P,
- a characteristic function $\lambda : F_i \longrightarrow \mathbb{Z}^n$.

Such manifolds possess an action of complexity $d = 0$ and the orbit space of such action is the polytope P.

Examples of complexity 1 actions

- The complex Grassmannian

$$Gr_{4,2}(\mathbb{C}) = \{V \subset \mathbb{C}^4 | \; dim V = 2\}.$$

  Its orbit space $Gr_{4,2}/T^3 \cong S^5$.
- The full complex flag manifold

$$Fl_3(\mathbb{C}) = \{V_\bullet = (\{0\} \subset V_1 \subset V_2 \subset \mathbb{C}^3) | \; dim V_i = i\}.$$

  Its orbit space $Fl_3/T^2 \cong S^4$.
- A quasitoric manifold $X^{2n}_{P,\lambda}$ has an action of a generic subtorus $T^{n-1} \subset T^n$. Its orbit space $X^{2n}_{P,\lambda}/T^{n-1} \cong S^{n+1}$.

All the examples we mentioned have orbit spaces homeomorphic to spheres. Is it normal?

The general position condition

Let $x \in M^T$, a fixed point, and consider the tangent representation of $T^k \curvearrowright T_x M^{2n}$; there is a decomposition: $T_x M^{2n} \cong V(\alpha_1) \oplus \ldots \oplus V(\alpha_n)$ where $V(\alpha_i)$ is the standard 1-dimensional complex representation given by $tz = \alpha_i(t)z, z \in \mathbb{C}$ and $\alpha_i \in Hom(T^k, S^1) \cong \mathbb{Z}^k$. The vectors α_i are called the weights of the tangent representation of T^k at $x \in M^T$.

Definition

The weights $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}^k$ are in j-general position (for $j \leq k$) if any j of them are linearly independent; the weights are in general position if they are in k-general position.

Definition

The action $T^k \curvearrowright M^{2n}$ is in general position if the weights $\alpha_1, \ldots, \alpha_n$ are in general position for any fixed point $x \in M^T$.

Local properties of complexity 1 actions

Consider an action of T^{n-1} on M^{2n} satisfying (*).

Proposition

• If the action $T^{n-1} \curvearrowright M^{2n}$ is in general position then the orbit space $Q = M^{2n}/T^{n-1}$ is a topological manifold. Otherwise the orbit space $Q = M^{2n}/T^{n-1}$ is a topological manifold with boundary $\partial Q \neq \emptyset$.

Therefore for the actions not in general position, the orbit space Q is not the sphere S^{n+1}.

Lemma

Let the tangent representation of T^{n-1} at $x \in M^T$ have the weights $\alpha_1, \ldots, \alpha_n$ such that $c_1\alpha_1 + \ldots + c_n\alpha_n = 0$ for $c_i \in \mathbb{Z} : gcd(c_i) = 1$. Then there is a neighbourhood U of x such that its orbit $TU \cong \mathbb{R}^{m+1} \times \mathbb{R}^{n-m}_{\geq 0}$ where m is the number of $c_i = 0$.

Hessenberg varieties

Definition

A function $h : [n] \longrightarrow [n]$ is called a Hessenberg function if it satisfies the following:

- $h(i) > i$ for $i = 1, \ldots, n-1$,
- $h(i+1) \geq h(i)$ for $i = 1, \ldots, n-1$.

Definition

Fix a linear transformation $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ and a Hessenberg function h. Consider

$$Hess(A, h) = \{(\{0\} \subset V_1 \subset \ldots \subset V_{n-1} \subset \mathbb{C}^n) |\ AV_i \subset V_{h(i)}\}$$

– a Hessenberg variety defined by A and h.

It follows from the definition that $Hess(A, h) \subset Fl_n(\mathbb{C})$.

Hessenberg varieties

We consider regular (nonsingular) Hessenberg varieties $H_h = Hess(\Lambda, h)$ where $\Lambda = diag(\lambda_1, \ldots, \lambda_n), \lambda_i \neq \lambda_j$. It has an effective T^{n-1}-action and its complexity

$$d = \sum_i (h(i) - i) - n + 1.$$

Proposition

For H_h with $d > 0$, the action is not in general position.

For $n = 4$, there are two Hessenberg functions, namely $h = (2, 4, 4, 4)$ and $h = (3, 3, 4, 4)$ such that $d = 1$. For the corresponding varieties, the following holds.

Example

The orbit space $H_h/T^3 \cong S^5 \setminus (\sqcup_{i=1}^4 D^5)$ – the complement of four disjoint open disks in the 5-sphere. In particular, $\partial(H_h/T^3) \cong \sqcup_{i=1}^4 S^4$.

Hamiltonian torus actions of complexity 1

Suppose now that the manifold M^{2n} is symplectic and the action $T^{n-1} \curvearrowright M^{2n}$ is Hamiltonian. In particular, there is a moment map $\mu : M^{2n} \longrightarrow \mathbb{R}^{n-1}$; its image $\mu(M^{2n}) = P^{n-1}$ is a convex polytope.

Theorem (Y.Karshon, S.Tolman '18)

If the action $T^{n-1} \curvearrowright M^{2n}$ is in general position then the orbit space M^{2n}/T^{n-1} is homeomorphic to the sphere S^{n+1}.

Corollary

If the action $T^{n-1} \curvearrowright M^{2n}$ is not in general position then the orbit space M^{2n}/T^{n-1} is homeomorphic to $S^{n+1} \setminus (U_1 \sqcup \ldots \sqcup U_m)$ where U_i are open domains.

In particular, one can find (co)homology groups of the orbit space using the Alexander Duality: $\tilde{H}^q(M^{2n}/T^{n-1}) = \tilde{H}_{n-q}(U_1 \sqcup \ldots \sqcup U_m)$.

Orbits spaces of Hessenberg varieties for n=5

In the case $n = 5$, there are three possible Hessenberg functions h for which H_h has complexity $d = 1$.

- $h = (2, 3, 5, 5, 5)$ and $h = (3, 3, 4, 5, 5)$. We have

$$H_h/T^4 \cong S^6 \setminus (\#_{K_5} D^6)$$

  where $\#_{K_5} D^6$ denotes the connected sum of five open disks D^6 along the full graph K_5 on five vertices. In particular, $\partial(H_h/T^4) \cong \#_{K_5} S^5$.

- $h = (2, 4, 4, 5, 5)$. We have

$$H_h/T^4 \cong S^6 \setminus (\#_{\tilde{K}_{5,5}} D^6)$$

  where $\tilde{K}_{5,5}$ denotes the almost full graph (see Fig.1). In particular, $\partial(H_h/T^4) \cong \#_{\tilde{K}_{5,5}} S^5$.

Almost full graph $\tilde{K}_{5,5}$

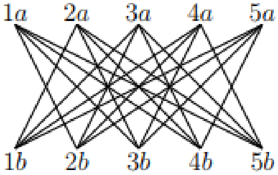

Figure: The almost full graph connecting two sets of vertices.

From the Alexander Duality it follows that:

$$H^4(Q_{(3,3,4,5,5)}) = H^4(S^6 \setminus (\#_{K_5} D^6)) \cong H_1(\#_{K_5} D^6) \cong \mathbb{Z}^6,$$

$$H^4(Q_{(2,4,4,5,5)}) = H^4(S^6 \setminus (\#_{\tilde{K}_{5,5}} D^6)) \cong H_1(\#_{\tilde{K}_{5,5}} D^6) \cong \mathbb{Z}^{11}$$

and $H^j(Q_{(3,3,4,5,5)}) = H^j(Q_{(2,4,4,5,5)}) = 0$ for $j \leq 3$.

Homologies of orbit spaces in general

Theorem (V.C. '19)

If the action $T^{n-1} \curvearrowright M^{2n}$ is Hamiltonian then the reduced cohomology groups $\tilde{H}^i(M^{2n}/T^{n-1}) = 0$ for $i = 0, 1, 2$.

Theorem (A. Ayzenberg, V.C. '19)

For a finite simplicial complex L, there is a smooth manifold M^{2n} with $H^{odd}(M^{2n}) = 0$ and an action of T^{n-1} such that $\tilde{H}_{q+3}(M^{2n}/T^{n-1}) = \tilde{H}_q(L)$ for $q \geq 0$. In fact, M^{2n}/T^{n-1} is homotopy equivalent to $\Sigma^3 L$.

Theorem (A. Ayzenberg, V.C. '19)

For a finite simplicial complex L, there is a smooth manifold M^{2n} with $H^{odd}(M^{2n}) = 0$ and an action of T^{n-1} in j-general position such that M^{2n}/T^{n-1} is homotopy equivalent to $\Sigma^{j+2} L$.

Further work

Theorem*

For a general action $T^{n-1} \curvearrowright M^{2n}$, the reduced cohomology groups $\tilde{H}^i(M^{2n}/T^{n-1}) = 0$ for $i = 0, 1, 2$.

Theorem*

If the action $T^{n-1} \curvearrowright M^{2n}$ is in j-position then $\tilde{H}^q(M^{2n}/T^{n-1}) = 0$ for $q < j + 2$.

Theorem*

More generally, if the action $T^k \curvearrowright M^{2n}$ is in j-position then $\tilde{H}^q(M^{2n}/T^k) = 0$ for $q < j + 2$.