Stratifications on generic torus orbit closures

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Toric Topology 2019, Okayama University of Science

Contents

1. Generic torus orbit closures (GTOCs) in T-varieties

- X : irreducible complex variety
- $T = (\mathbb{C}^*)^r$ $(r \geq 1)$: algebraic torus of rank r
- T acts on X algebraically
- X^T : the set of T-fixed points in X.
  We assume that $|X^T| < \infty$

Definition

A point $x \in X$ is called **T-generic point** if $X^T \subset \overline{T \cdot x}$

The condition

$$X^T \subset \overline{T \cdot x}$$

is seemingly very extremal.

$\longrightarrow$ Is this condition really a "generic condition" ?

Examples

- Consider the action $\mathbb{C}^* \curvearrowright \mathbb{C}P^2$ given by

$$t \cdot [x_0 : x_1 : x_2] := [x_0 : tx_1 : t^2 x_2].$$

  There is <u>no</u> T-generic point.

- Consider the action $\mathbb{C}^* \curvearrowright \mathbb{C}^2$ given by

$$t \cdot (x, y) := (tx, t^{-1} y).$$

  The origin $(0, 0)$ is a <u>unique</u> T-generic point.

Lemma (Y.)

Assume that X is T-equivariantly embedded to a T-variety $\underline{X}$ satisfying the following conditions:

- $\underline{X}$ has a T-invariant open covering $\{U_p\}_{p \in \underline{X}^T}$ s.t. each U_p is T-isomorphic to some rational complex T-representation V_p.
- The weights $\alpha_1, \ldots, \alpha_n : T \to \mathbb{C}^*$ of V_p satisfy

$$\langle \mathbf{v}, \alpha_i \rangle > 0 \quad (i = 1, \ldots, n)$$

  for some $\mathbf{v} \in \mathrm{Hom}(T, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$.
- X is covered by $\{U_p\}_{p \in X^T}$.

Then the set of T-generic points of X is given by $\cap_{p \in X^T}(X \cap U_p)$. In particular T-generic points form a Zariski dense subset in X.

Remark

- A point $x \in X$ is called **regular point** if

$$\dim T \cdot x = \max\{\dim T \cdot y | y \in X\}.$$

  Regular points also form a non-empty Zariski open set.
- But "regular" and "T-generic" are not equivalent in general: Let

$$X := GL_3(\mathbb{C})/B, \quad g := \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad x := gB.$$

  Then x is regular ($\because$ $T \cdot x$ is a free orbit) but not T-generic.

Example

- G : complex reductive algebraic group
- B : Borel subgroup, B^- : opposite Borel subgroup
- W : Weyl group
- $X_w = \overline{BwB/B}$, $X^u = \overline{B^- uB/B}$: (opposite) Schubert variety
- $X_w^u := X_w \cap X^u$: Richardson variety

$X = X_w^u$ satisfies the assumptions in above Lemma ($\underline{X} = G/B$).
$\longrightarrow X_w^u$ has T-generic points.

Since $X_w = X_w^{id}$, Schubert variety X_w has T-generic points.

2. The case of Schubert varieties of type A (Lee-Masuda's work)

In the rest of this talk we focus on Schubert variety X_w of type A. ($G = GL_n(\mathbb{C}), B =$ (upper triangular Borel))

In the paper

Generic torus orbit closures in Schubert varieties, J. of Comb. Series. A, 2019,

Lee-Masuda investigate GTOCs Y_w in X_w.

Every torus orbit closure in $Fl(\mathbb{C}^n)$ is normal by Carrell-Kurth, Thus Y_w is a normal toric variety. Y_w can be singular.

Lee-Masuda obtain:

- Explicit description of the fan of Y_w.
- Combinatorial criterion for the smoothness of Y_w
- Some interesting conjectures which indicate algebro-geometric similarity between X_w and Y_w.

Notation

Let $u, w \in \mathfrak{S}_n, u \leq w$ and $t_{u(i),u(j)}u$ be the transposition

$$u = \cdots u(i) \cdots u(j) \cdots \Longrightarrow t_{u(i),u(j)}u = \cdots u(j) \cdots u(i) \cdots$$

- Let $\widetilde{E}_w(u)$ be the set of pairs $(u(i), u(j))$ s.t.
  (1) $1 \leq i < j \leq n$
  (2) $t_{u(i),u(j)}u \leq w$
  (3) $|\ell(t_{u(i),u(j)}u) - \ell(u)| = 1$
- $(a, b) \in \widetilde{E}_w(u)$ is said to be **indecomposable** if there is no sequence

$$(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k) \in \widetilde{E}_w(u)$$

  s.t. $k \geq 2, a = a_1, b_1 = a_2, \ldots, b_{k-1} = a_k, b_k = b.$
- $E_w(u) :=$ (the set of indecomposables in $\widetilde{E}_w(u)$)

Description of dual cones

Note that

$$Y_w^T = X_w^T = \{u \in \mathfrak{S}_n | u \leq w\}.$$

We denote by $C_w(u)$ the maximal cone corresponding to $u \in Y_w^T$.

Theorem (Lee-Masuda)

The set of primitive edge vectors of the dual cone

$$D_w(u) := (C_w(u))^\vee$$

is given by

$$\{\mathbf{e}_b - \mathbf{e}_a | (a, b) \in E_w(u)\}.$$

Remark

(1) GTOCs in X_w are mutually isomorphic as T-varieties (this justifies the notation "Y_w").

This kind of "invariance" is also true for GTOCs in flag Bott manifolds (Lee-Suh's work. Explicit description of fans).

(2) Question:

Does the invariance also hold in the setting of Section 1 ?

If the invariance holds for X, it follows that **every T-generic point in X is a regular point in X.**

3. Main result

Theorem (Y., a conjecture of Lee-Masuda)

The Poincaré polynomial of Y_w is given by

$$P_t(Y_w) = \sum_{u \in \mathfrak{S}_n, u \leq w} t^{a_w(u)}$$

where

$$a_w(u) := |\{(a, b) \in E_w(u) | a < b\}|$$

4. Sketch of proof

Step 1: Local description of $\mathbf{Y_w}$

Recall that $D_w(u)$ is the dual cone of the maximal cone $C_w(u)$ corresponding to $u \in Y_w^T$.

Key Lemma

The monoid $D_w(u) \cap \mathbb{Z}^n$ of lattice points in $D_w(u)$ is generated by primitive edge vectors.

$\longrightarrow$ The affine variety

$$Y_w(u) := \mathrm{Max}(\mathbb{C}[D_w(u) \cap \mathbb{Z}^n]) \quad (T\text{-inv. open nbd around } u \in Y_w)$$

can be realized as an algebraic set in $\mathbb{C}^{E_w(u)}$.

Step 2: Defining equations

One can show that the defining equations of

$$Y_w(u) \quad (\subset \mathbb{C}^{E_w(u)})$$

are given by

$$\prod_{(a,b)\in L} x_{(a,b)} = \prod_{(a,b)\in R} x_{(a,b)}$$

where

- $L, R \subset E_w(u)$
- $\displaystyle\sum_{(a,b)\in L} (\mathbf{e}_b - \mathbf{e}_a) = \sum_{(a,b)\in R} (\mathbf{e}_b - \mathbf{e}_a)$

Step 3: Computing BB-cell

Thanks to the defining equations, one can analyze the BB-cell $Y_w^+(u)$ with respect to the $\mathbb{C}^*$-action on $Y_w(u)$ induced from the regular cocharacter

$$\mathbb{C}^* \to T, t \mapsto (t, t^2, \ldots, t^n):$$

Theorem (Y.)

(1) $Y_w^+(u) \cong \mathbb{C}^{E_w^<(u)}$

(2) $\{Y_w^+(u)\}_{u \in Y_w^T}$ is an affine paving of Y_w.

Theorem in Section 3 follows from above theorem.

Summary

- Concept of a T-generic point in a T-variety:
  1. T-generic points form a Zariski dense subset under some mild conditions
  2. Question on uniqueness of T-isomorphic types of GTOCs in a T-variety
- GTOCs in the Schubert variety X_w of type A
  1. Lee-Masuda's conjecture on the Poincaré polynomials of GTOCs in X_w
  2. Paving theorem

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Thank you for your attention !