Generic torus orbit closures in Richardson varieties

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Based on joint work with Eunjeong Lee (IBS-CGP) and Mikiya Masuda (OCU)

Let X be a complex projective algebraic variety having an action of algebraic torus $\mathbb{T} = (\mathbb{C}^*)^n$.

If the action of $\mathbb{T}$ extends to a linear action on the ambient projective space $\mathbb{P}^N$, then we get a moment map $\mu\colon X \hookrightarrow \mathbb{P}^N \to \mathbb{R}^n$. Furthermore, for each point $x \in X$, the image $\mu(\overline{\mathbb{T}x})$ is a rational convex polyhedron in $\mathbb{R}^n$.

A point $x \in X$ is *generic* ($\mathbb{T}$-*generic*) if $X^{\mathbb{T}} = \overline{\mathbb{T}x}^{\mathbb{T}}$.

If $\dim_{\mathbb{C}} X = d$ and the action of $\mathbb{T}$ on X is effective, we call the number $d - n$ the complexity of the action.

In this talk, X is a Richardson variety in the flag manifold $\mathcal{F}\ell_n$ and we study the topology of X using the combinatorics of $\mu(\overline{\mathbb{T}x})$ for a generic point $x \in X$ when X has the torus action of complexity ≤ 1.

Let B be the set of all upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$. Then $\mathrm{GL}_n(\mathbb{C})/B$ is a *flag manifold* and denoted by $\mathcal{F}\ell_n$. i.e.,

$$\mathcal{F}\ell_n = \{gB \mid g \in \mathrm{GL}_n(\mathbb{C})\}.$$

Let $\mathfrak{S}_n$ be the permutation group on $\{1, 2, ..., n\}$. For $w \in \mathfrak{S}_n$, we write

$$w = w(1)w(2)\cdots w(n) \text{ or } w = [w(1), \ldots, w(n)].$$

Set $e = [1, 2, \ldots, n]$ and $w_0 = [n, n-1, \ldots, 1]$.
Then for each $w \in \mathfrak{S}_n$, we get

$$w = \begin{bmatrix} | & | & & | \\ \mathbf{e}_{w(1)} & \mathbf{e}_{w(2)} & \cdots & \mathbf{e}_{w(n)} \\ | & | & & | \end{bmatrix} \in \mathrm{GL}_n(\mathbb{C}),$$

and wB is called a *complete coordinate flag*.

Let $\mathbb{T}$ be the set of all diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$:

$$\mathbb{T} = \left\{ \begin{bmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{bmatrix} \in \mathrm{GL}_n(\mathbb{C}) \right\} \cong (\mathbb{C}^*)^n.$$

Then $\mathbb{T}$ is a maximal torus in B and it acts on $\mathcal{F}\ell_n$:

$$t \cdot gB := (tg)B \text{ for } t \in \mathbb{T} \text{ and } g \in \mathrm{GL}_n(\mathbb{C}).$$

The $\mathbb{T}$-fixed point set of $\mathcal{F}\ell_n$ is

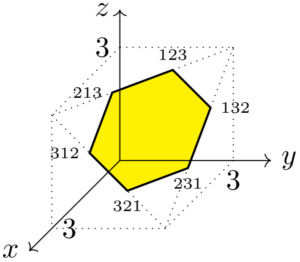
$$(\mathcal{F}\ell_n)^{\mathbb{T}} = \{wB \mid w \in \mathfrak{S}_n\}.$$

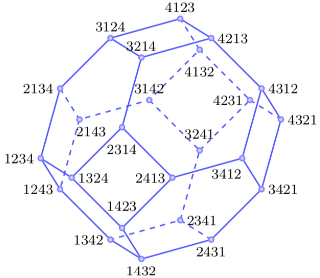
Using the Plücker embedding, we get a moment map

$$\mu\colon \mathcal{F}\ell_n \longrightarrow \mathbb{R}^n$$

such that $\mu(wB) = (w^{-1}(1), w^{-1}(2), \cdots, w^{-1}(n))$, and hence the image of μ is the permutohedron

$$\mathrm{Perm}_{n-1} := \mathrm{ConvHull}\{(w(1), \ldots, w(n)) \in \mathbb{R}^n \mid w \in \mathfrak{S}_n\}.$$





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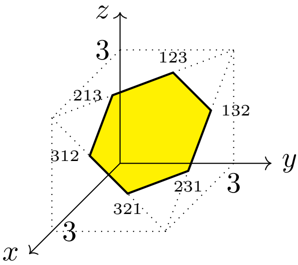

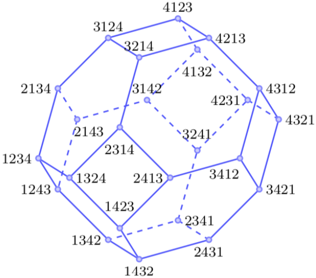

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Let B^- be the set of all lower triangular matrices in $\mathrm{GL}_n(\mathbb{C})$. Then we have

$$\mathrm{GL}_n(\mathbb{C}) = \coprod_{w \in \mathfrak{S}_n} BwB = \coprod_{w \in \mathfrak{S}_n} B^- wB.$$

Hence, we get

$$\mathcal{F}\ell_n = \coprod_{w \in \mathfrak{S}_n} BwB/B = \coprod_{w \in \mathfrak{S}_n} B^- wB/B.$$

Note that

$$BwB/B \cong \mathbb{C}^{\ell(w)} \text{ and } B^- wB/B \cong \mathbb{C}^{\ell(w_0) - \ell(w)},$$

where $\ell(w)$ is the number of inversions of w, i.e.,

$$\ell(w) = \#\{(i,j) \mid 1 \leq i < j \leq n \text{ and } w(i) > w(j)\}.$$

For each $w \in \mathfrak{S}_n$, the Schubert variety X_w and the opposite Schubert variety X^w are defined as

$$X_w := \overline{BwB/B} \quad \text{and} \quad X^w := \overline{B^- wB/B},$$

respectively. Then

$$X_w = \coprod_{v \le w} BwB/B \quad \text{and} \quad X^w = \coprod_{v \ge w} B^- wB/B,$$

where $v \le w$ if and only if $(v(1), \dots, v(i))\Uparrow \le (w(1), \dots, w(i))\Uparrow$ for $1 \le \forall i \le n$.* Then $(\mathfrak{S}_n, \le)$ is a poset and the partial ordering $\le$ is the Bruhat order.

For $v \le w$ in $\mathfrak{S}_n$, the Richardson variety X_w^v is defined as

$$X_w^v := X_w \cap X^v.$$

*$(v(1), \dots, v(i))\Uparrow$ stands for "reordered to increasing order".

Each Richardson variety X_w^v is a $\mathbb{T}$-invariant irreducible subvariety of $\mathcal{F}\ell_n$ and

$$(X_w^v)^{\mathbb{T}} = \{uB \mid v \leq u \leq w\}.$$

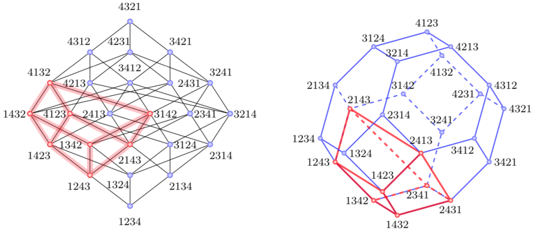
Recall that the moment map $\mu\colon \mathcal{F}\ell_n \to \mathbb{R}^n$ satisfies

$$\mu(wB) = (w^{-1}(1), w^{-1}(2), \cdots, w^{-1}(n)).$$

Hence

$$\mu(X_w^v) = \mathrm{ConvHull}\{(u^{-1}(1), \ldots, u^{-1}(n)) \mid v \leq u \leq w\}.$$

(e.g.) The moment map image of X_{4132}^{1243}



Note that for a point $x \in X_w^v$, since $\overline{\mathbb{T}x} \subseteq X_w^v$, we have $(\overline{\mathbb{T}x})^{\mathbb{T}} \subseteq (X_w^v)^{\mathbb{T}}$.
We say that a point $x \in X_w^v$ is *generic* if $(\overline{\mathbb{T}x})^{\mathbb{T}} = (X_w^v)^{\mathbb{T}}$.
We define

$$\begin{aligned} c(v,w) &= \dim_{\mathbb{C}} X_w^v - \dim_{\mathbb{C}} \overline{\mathbb{T}x} \\ &= \dim_{\mathbb{C}} X_w^v - \dim_{\mathbb{R}} \mu(X_w^v), \end{aligned}$$

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A Richardson variety X_w^v is a toric variety if and only if $c(v,w) = 0$. In this case, the Richardson variety X_w^v is the toric variety defined by the polytope $\mu(X_w^v)$.

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Today

We are interested in the smooth Richardson varieties X_w^v with $c(v, w) \leq 1$:

1. X_w^v with $c(v, w) = 0$, and
2. X_w with $c(e, w) = 1$.

Schubert variety	$\underset{\text{desingularize}}{\rightsquigarrow}$	Bott-Samelson variety	$\underset{\text{diffeo.}}{\approx}$	Bott manifold
X_w		$Z_{\underline{w}}$		B

For $w = s_{i_1} \dots s_{i_r}$, let $P_{i_k} = \overline{B s_{i_k} B}$. Then $Z_{\underline{w}}$ be the quotient:

$$Z_{\underline{w}} := (P_{i_1} \times \cdots \times P_{i_r}) / B^r,$$

with respect to the action of $B^r := \underbrace{B \times \cdots \times B}_{r}$ by

$$(p_1, \dots, p_r) \cdot (b_1, \dots, b_r) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{r-1}^{-1} p_r b_r)$$

for $(p_1, \dots, p_r) \in \prod_{k=1}^{r} P_{i_k}$ and $(b_1, \dots, b_r) \in B^r$. Then $Z_{\underline{w}}$ is a smooth projective variety, but not a toric variety in general.

A *Bott tower* is an iterated $\mathbb{C}P^1$-bundle:

$$B_{2n} = P(\underline{\mathbb{C}} \oplus \xi_{n-1}) \xrightarrow{\mathbb{C}P^1} B_{2(n-1)} \xrightarrow{\mathbb{C}P^1} \cdots \xrightarrow{\mathbb{C}P^1} B_2 = \mathbb{C}P^1 \xrightarrow{\mathbb{C}P^1} \{\text{a point}\},$$

where each B_{2k} is the complex projectivization of the Whitney sum of a complex line bundle ξ_{k-1} over $B_{2(k-1)}$ and the trivial bundle $\underline{\mathbb{C}}$.
Each B_{2k} is called a *Bott manifold* (of height k), and it is a projective smooth toric variety.

Theorem [Fan (1998), Karuppuchamy (2013)]

The following are equivalent.

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Recall that

- $(X_w^v)^{\mathbb{T}} = \{uB \mid v \leq u \leq w\}$ and
- $\mu(X_w^v) = \mathrm{ConvHull}\{(u^{-1}(1), \ldots, u^{-1}(n)) \mid v \leq u \leq w\}$.

Bruhat interval

$$[v, w] := \{z \in \mathfrak{S}_n \mid v \leq z \leq w\}$$

Kodama and Williams (2013) define the Bruhat interval polytope

$$Q_{v,w} := \mathrm{ConvHull}\{(z(1), \ldots, z(n)) \mid v \leq z \leq w\}$$

for $v \leq w$ in $\mathfrak{S}_n$.

Therefore $\mu(X_w^v) = Q_{v^{-1}, w^{-1}}$.

We call $Q_{v,w}$ toric if the Richardson variety $X_{w^{-1}}^{v^{-1}}$ is a toric variety.

Note that

- $[v, w] \cong [v^{-1}, w^{-1}]$ (i.e., $[x, y] \subset [v, w] \Leftrightarrow [x^{-1}, y^{-1}] \subset [v^{-1}, w^{-1}]$)
- $Q_{v,w}$ and $Q_{v^{-1},w^{-1}}$ are not combinatorially equivalent in general.
  - If $n \leq 4$, then $Q_{v,w} \cong Q_{v^{-1},w^{-1}}$ for $v \leq w$ in $\mathfrak{S}_n$.
  - If $n > 4$, then there are many examples that $Q_{v,w} \ncong Q_{v^{-1},w^{-1}}$.

Example

Note that $35412^{-1} = 45132$, and $\ell(35412) = 7$.
The Bruhat interval polytopes $Q_{e,35412}$ and $Q_{e,45132}$ are 5-dim'l.
The face vectors of $Q_{e,35412}$ and $Q_{e,45132}$ are

$$f(Q_{e,35412}) = (1, 60, 123, 82, 19, 1)$$
$$f(Q_{e,45132}) = (1, 60, 122, 81, 19, 1).$$

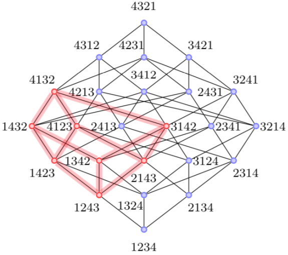
Therefore, $Q_{e,35412}$ and $Q_{e,45132}$ are not combinatorially equivalent.

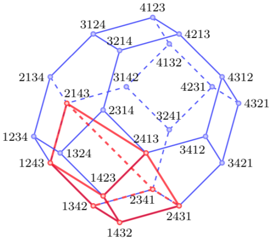
Theorem [Lee-Masuda-Park]

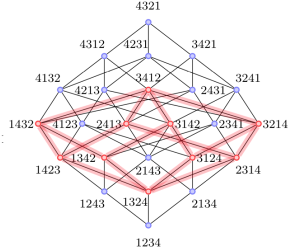
1. $\dim_{\mathbb{R}} Q_{v,w} = \dim_{\mathbb{R}} Q_{v^{-1},w^{-1}}$ and hence $c(v,w) = c(v^{-1}, w^{-1})$.
2. $Q_{v,w}$ is toric if and only if $Q_{x,y}$ is a face of $Q_{v,w}$ for every $[x,y] \subseteq [v,w]$.
3. $Q_{v,w}$ is smooth if and only if it is simple.

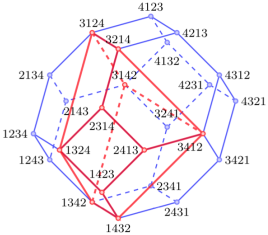
Therefore, X^v_w is a smooth projective toric variety if and only if $c(v,w) = 0$ and $\mu(X^v_w)$ is a simple polytope.

Examples: X_{2431}^{1243} and X_{3421}^{1324} are toric.









Every toric Schubert variety is smooth, but not every toric Richardson variety is.

Proposition [Lee-Masuda-Park]

Assume $Q_{v,w}$ is toric. Then $Q_{v,w}$ is simple if and only if it is a cube.

Theorem [Lee-Masuda-Park]

The following are equivalent:

1. X^v_w is a smooth toric variety.
2. $c(v,w) = 0$ and $[v,w]$ is Boolean.
3. $Q_{v,w}$ is a cube.
4. X^v_w is a Bott manifold.

A Richardson variety X^v_w is a Bott manifold if and only if it is toric and $[v,w]$ is Boolean.

A simple transposition is a permutation of the form

$$s_i = [1, \ldots, i-1, i+1, i, i+2, \ldots, n] \qquad (1 \le i \le n-1).$$

Every $w \in \mathfrak{S}_n$ can be expressed as a product of simple transpositions. A minimal length expression of w is said to be reduced.

For $w \in \mathfrak{S}_n$, $\dim_{\mathbb{R}} Q_{e,w}$ is the number of distinct letters appearing in a reduced expression of w.

For example,

1. $321 = s_1 s_2 s_1 = s_2 s_1 s_2$ and hence $c(e, 321) = 3 - 2 = 1$
2. $3412 = s_2 s_3 s_1 s_2 = s_2 s_1 s_3 s_2$ and hence $c(e, 3412) = 4 - 3 = 1$.

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Let $w \in \mathfrak{S}_n$ and $p \in \mathfrak{S}_k$ for $k \leq n$.The permutation w contains the pattern p if there exist $i_1 < \cdots < i_k$ such that $w(i_1) \cdots w(i_k)$ is in the same relative order as $p(1) \cdots p(k)$. If w does not contains p, then w avoids p, or is p-avoiding.

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Let $w \in \mathfrak{S}_n$. Then

1. $c(e, w) = 0$ if and only if $[321; 3412](w) = 0$.
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A Schubert variety X_w is smooth and has complexity one if and only if w avoids 3412 and contains the patter 321 exactly once.

(e.g.) The permutations 321, 4132, 4213, 2431, and 3241 give smooth Schubert varieties of complexity 1.

Using the result of Tenner (2012), we get the following:

If a permutation w avoids every pattern in the set $\{3412, 4231, 4321\}$, then the Schubert variety X_w is smooth and the complexity of X_w is the number of distinct 321-patterns in w.

A Schubert variety X_w is smooth and has complexity one if and only if w avoids 3412 and contains the patter 321 exactly once.

(e.g.) The permutations 321, 4132, 4213, 2431, and 3241 give smooth Schubert varieties of complexity 1.

Using the result of Tenner (2012), we get the following:

If a permutation w avoids every pattern in the set $\{3412, 4231, 4321\}$, then the Schubert variety X_w is smooth and the complexity of X_w is the number of distinct 321-patterns in w.

Proposition [Lee-Masuda-Park]

Let w be a permutation in $\mathfrak{S}_n$ containing exactly one 321 pattern and avoiding 3412. Then

1. the Bruhat interval $[e, w]$ is isomorphic to a poset $\mathfrak{S}_3 \times B_{\ell-3}$, where $\ell = \ell(w)$ and $B_{\ell-3}$ is the Boolean poset of length $(\ell - 3)$, and
2. the Bruhat interval polytope $Q_{v,w}$ is combinatorially equivalent to the polytope $\mathrm{Perm}_2 \times I^{\ell-3}$, where Perm_2 is the two-dimensional permutohedron (i.e., hexagon).

Theorem [Lee-Fujita-Suh]

Schubert variety	$\underset{\text{desingularize}}{\rightsquigarrow}$	flag Bott-Samelson variety	$\underset{\text{diffeo.}}{\approx}$	flag Bott manifold
X_w		$Z_{\mathcal{I}}$		B

Theorem [Lee-Masuda-Park]

For $x \in \mathfrak{S}_n$, if w avoids 3412 and has the pattern 321 exactly once, then X_w is isomorphic to a flag Bott-Samelson variety, and hence it is diffeomorphic to a flag Bott manifold.

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It follows from the fact that w avoids 3412 and has the pattern 321 exactly once if and only if there exists a reduced expression of w containing $s_i s_{i+1} s_i$ as a factor and no other repetitions. See [Daly(2013)].

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Thank you very much!