On orbit braids

(Joint work with Hao Li and Fengling Li)

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Outline

- History of braid groups
- Orbit braid group
- Main Results
- Calculations of orbit braid groups

History of braid groups

- **E. Artin first defined the braids and braid groups in 1925**
  [Theorie der Zöpfe, *Abh. Math. Sem. Univ. Hamburg* **4** (1925), 47–72]
  [Theory of braids, *Ann. of Math.* **48** (1947), 101–126]

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  [Theory of braids, *Ann. of Math.* **48** (1947), 101–126]

- **Roots of the notion can be seen in the researches of the following**
  Hurwitz: [Über Riemannsche Flächen mit gegebenen Verzweigungspunkren, *Math. Ann.* **39** (1891), 1–61]
  Fricke–Klein: [Vorlesungen über die Theorie der automorphen Funktionen, Bd. I. *Gruppentheoretischen Grundlagen*, Teuner, Leipzig 1897]
  Even in Gauss's notebook

Development of the theory of braids

After the work of Artin, the subject has continued to further develop by extending ideas of braid groups or combining with various ideas and theories from other research areas.

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- Alternative description using the fundamental groups of configuration spaces (Fox and Neuwrith).
- Generalized braid groups by Brieskorn to all finite Coxeter groups.
- Applications in low-dimensional topology, especially in the study of links and knots. E.g., a vast family of link invariants were constructed using braids.
- Cohomology theory of braid groups (E.g., Arnol'd's work)
- Connection with other theories, such as theory of free groups
- Connection with other areas, such as Chern-Simons perturbation theory in mathematical physics and Yang-Baxter equation in physics

Original point of view for braids

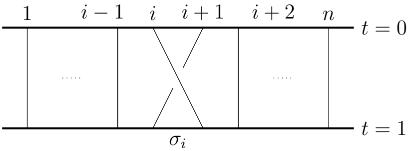
Original viewpoint of Artin

Braids arise naturally as isotopy classes of a collection of n connected strings in three-dimensional space $\mathbb{R}^2 \times I$.

Theorem (Artin)

Br_n is generated by $\sigma_i, i = 1, ..., n-1$ with the relations

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for} \quad |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{cases}$$



Alternative description of braid groups

In 1962, E. Fadell and L. Neuwirth introduced the notion of configuration spaces

Definition

For a topological space X, the **configuration space** of X at n points is defined as follows:

$$F(X, n) = \{(x_1, \dots, x_n) \in X^{\times n} | x_i \neq x_j \quad \forall i \neq j\}$$

with subspace topology where $n \geq 2$.

Remark. The notion of configuration space first appeared in physics in the 1940s

Alternative description of braid groups

Meanwhile, R. Fox and L. Neuwirth showed in 1962 that

Theorem (Fox–Neuwirth)

$$Br_n \cong \pi_1(F(\mathbb{R}^2, n)/\Sigma_n)$$

Remark Recently, people often define the braid groups from the viewpoint of configuration spaces.

Let M be a connected top. manifold of dim > 1. Symmetric group $\Sigma_n \curvearrowright F(M, n)$ freely.

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- $1 \longrightarrow P_n(M) \longrightarrow B_n(M) \longrightarrow \Sigma_n \longrightarrow 1$

An explanation in the general case

In the viewpoint of Artin,

Braids in $M \times I$ can be constructed by using paths in $F(M, n)$

• Given a path $\alpha = (\alpha_1, ..., \alpha_n) : I \longrightarrow F(M, n)$ with $\alpha(0) = \mathbf{x}$ and $\alpha(1) = \mathbf{x}_\sigma \in \Sigma_n(\mathbf{x})$.

• α gives a braid $c(\alpha) = \{c(\alpha_1), ..., c(\alpha_n)\}$ of n strings in $M \times I$, where each string $c(\alpha_i) = \{(\alpha_i(s), s) | s \in I\} \approx I$.

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Equivalence

Let $\alpha, \beta : I \longrightarrow F(M, n)$ be two paths with the same endpoints. Then $\alpha \simeq \beta(\text{rel } \partial I) \Longleftrightarrow c(\alpha) \sim_{isotopy} c(\beta)$ in $M \times I$.

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Theorem

$$B_n(M) \cong \pi_1(F(M, n)/\Sigma_n)$$

Generalized braid groups

In 1970's, Brieskorn generalized the concept of classical braid group from symmetric group to all finite Coxeter groups, which is called generalized braid group or Artin group.

Let

$$W = \langle w_1, \ldots, w_k \big| w_i^2 = e, (w_i w_j)^{m_{ij}} = e \rangle$$

be a finite Coxeter group where $m_{ij} = m_{ji}$.

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Definition of generalized braid group

The **generalized braid group** $Br(W)$ of W is defined as the group with generators w_i and relations

$$\mathrm{prod}(m_{ij}; w_i, w_j) = \mathrm{prod}(m_{ji}; w_j, w_i)$$

where the symbol $\mathrm{prod}(m; x, y)$ stands for the product $xyxy\cdots$ with m factors.

Geometric realization of generalized braid groups

First

- V: an n-dim real vector space
  W: considered as a finite subgroup of $GL(V)$ generated by reflections
  $\mathcal{M}$: the set of hyperplanes such that W is generated by the orthogonal reflections in the $M \in \mathcal{M}$, and assume that $w(M) \in \mathcal{M}$ for any $w \in W$ and any $M \in \mathcal{M}$.

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Next

- consider the complexification $V_{\mathbb{C}}$ of V and the complexification $M_{\mathbb{C}}$ of $M \in \mathcal{M}$.
- Set $Y_W = V_{\mathbb{C}} - \bigcup_{M \in \mathcal{M}} M_{\mathbb{C}}$
- W acts freely on Y_W, so we have the quotient $X_W = Y_W / W$.
- $1 \longrightarrow \pi_1(Y_W) \longrightarrow \pi_1(X_W) \longrightarrow W \longrightarrow 1.$

Geometric realization of generalized braid groups

Theorem (Brieskorn–Deligne)

(1) $\pi_1(X_W) \cong Br(W)$;
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Remark

- Generalized braid group $Br(W)$ is realized by the fundamental group $\pi_1(X_W)$
- The fundamental group $\pi_1(Y_W)$ is called the pure braid group, also denoted by $P(W)$.
- $1 \longrightarrow P(W) \longrightarrow Br(W) \longrightarrow W \longrightarrow 1.$

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- Choose a connected topological manifold $\mathbb{M}$ admitting an action of a finite group $\mathbb{G}$.
- Let $Y_{\mathbb{G}}$ be formed by all points of free orbit type in $\mathbb{M}$. So the action of $\mathbb{G}$ restricted to $Y_{\mathbb{G}}$ is free. Assume that $Y_{\mathbb{G}}$ is connected. Then there is a fibration $Y_{\mathbb{G}} \longrightarrow X_{\mathbb{G}}$ with fiber $\mathbb{G}$, which gives a short exact sequence:

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- The fundamental group $\pi_1(X_{\mathbb{G}})$ is called the **braid group** of the action of $\mathbb{G}$ on $\mathbb{M}$, denoted by $Br(\mathbb{M}, \mathbb{G})$, and the fundamental group $\pi_1(Y_{\mathbb{G}})$ is called the **pure braid group** of the action of $\mathbb{G}$ on $\mathbb{M}$, denoted by $P(\mathbb{M}, \mathbb{G})$.

Motivation and Aim

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Our strategy

Our strategy to do this is to mix the original idea of Artin and the theory of transformation groups together by making use of the construction of orbit configuration spaces.

Orbit configuration space

Definition (M. A. Xicoténcatl, Thesis (Ph.D.)-University of Rochester. 1997)

Given a topological group G and a topological space X with an effective G-action. Then the **orbit configuration space** of the G-space X is defined by

$$F_G(X,n) = \{(x_1,\dots,x_n) \in X^n \mid G(x_i) \cap G(x_j) = \emptyset \quad \text{for } i \neq j\}$$

with subspace topology, where $n \geq 2$ and $G(x)$ denotes the orbit of x.

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Remark: Pay our attention on the case:
G: **a finite group**
X: **a connected topological manifold M of dim > 1 with an effective G-action.**
So $F_G(M, n)$ is connected.

Orbit braids

Fix a point $\mathbf{x} = (x_1, ..., x_n) \in F_G(X, n)$ as a base point where the orbit $G(x_i)$ at x_i is of free type.

Let $\mathbf{x}_\sigma = (x_{\sigma(1)}, ..., x_{\sigma(n)})$, $\sigma \in \Sigma_n$.

Braids in $M \times I$ from paths in $F_G(M, n)$

- Take a path $\alpha : I \longrightarrow F_G(M, n)$ with $\alpha(0) = \mathbf{x}$ and $\alpha(1) = \mathbf{x}_\sigma$.

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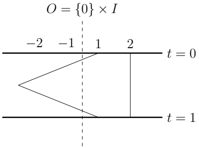
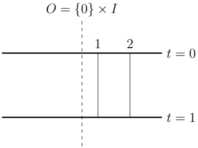
Remark. If we forget the action of G on M, then $c(\alpha)$ becomes a braid in the sense of Artin. Otherwise, $c(\alpha)$ would be different from the classical one.

Example

Consider the orbit configuration space $F_{\mathbb{Z}_2}(\mathbb{C}, n)$ where the action of $\mathbb{Z}_2$ on $\mathbb{C}$ is given by $z \longmapsto -z$, so this action is non-free and fixes only the origin of $\mathbb{C}$. In the case of $n = 2$, let us see two closed paths $\alpha, \beta : I \longrightarrow F_{\mathbb{Z}_2}(\mathbb{C}, 2)$ at the point $\mathbf{x} = (1, 2)$ such that their corresponding braids $c(\alpha)$ and $c(\beta)$ are as shown below:

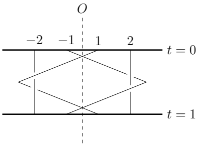


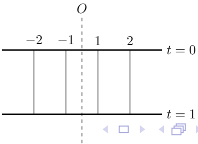
$c(\alpha)$



$c(\beta)$

If we forget the action of $\mathbb{Z}_2$ on $\mathbb{C}$, then clearly $c(\alpha)$ and $c(\beta)$ are isotopic relative to endpoints in $\mathbb{C} \times I$. However, under the condition that $\mathbb{C}$ admits the action of $\mathbb{Z}_2$, both $c(\alpha)$ and $c(\beta)$ are not isotopic since the first string of $c(\alpha)$ cannot go through the orbit of the second string of $c(\alpha)$, as we can see from the following left picture.





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Let $\alpha = (\alpha_1, ..., \alpha_n) : I \longrightarrow F_G(M, n)$ be a path such that $\alpha(0) = \mathbf{x}$ and $\alpha(1) = g\mathbf{x}_\sigma$ for some $(g, \sigma) \in G^{\times n} \times \Sigma_n$. Then

$$\widetilde{c(\alpha)} = \{\widetilde{c(\alpha_1)}, ..., \widetilde{c(\alpha_n)}\}$$

is called an **orbit braid** in $M \times I$,

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Fix $\widetilde{c(\mathbf{x})} = \{G(x_1), ..., G(x_n)\}$ as an **orbit base point.**
Natural operation:

$$\widetilde{c(\alpha)} \circ \widetilde{c(\beta)}|_{s \in I} = \begin{cases} \widetilde{c(\alpha)}|_{2s \in I} & \text{if } s \in [0, \frac{1}{2}] \\ \widetilde{c(\beta)}|_{2s-1 \in I} & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

but this operation is not associative

Equivalence relation among ordinary braids

Recall

Equivalence in the theory of classical braids

Let $\alpha, \beta : I \longrightarrow F(M, n)$ be two paths with the same endpoints.

$$\alpha \simeq \beta (\mathrm{rel}\ \partial I) \Longleftrightarrow c(\alpha) \sim_{iso} c(\beta).$$

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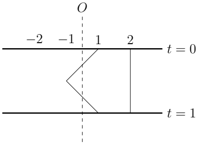
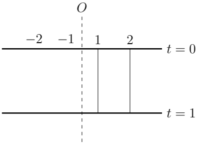
- In the theory of ordinary braids, isotopy is used as the equivalence relation among ordinary braids.
- However, equivariant isotopy is not sufficient enough to be used as the equivalence relation among orbit braids.

Example

Let the action of $\mathbb{Z}_2$ on $\mathbb{C}$ be the same as the above example. Consider the orbit configuration space $F_{\mathbb{Z}_2}(\mathbb{C}, n)$. In the case of $n = 2$, take two closed paths $\alpha, \beta : I \longrightarrow F_{\mathbb{Z}_2}(\mathbb{C}, 2)$ at the base point $\mathbf{x} = (1, 2)$ such that their corresponding ordinary braids $c(\alpha)$ and $c(\beta)$ are shown as follows:

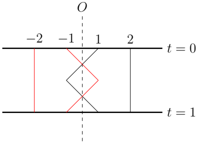


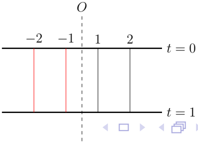
$c(\alpha)$



$c(\beta)$

Clearly, $c(\alpha) \sim^{G}_{iso} c(\beta)$. This means that orbit braids $\widetilde{c(\alpha)}$ and $\widetilde{c(\beta)}$ as shown below are essentially the same in such a sense that the first string of $c(\alpha)$ can be deformed into the first string of $c(\beta)$ in $M \times I$ under the action of G. However, $\widetilde{c(\alpha)}$ and $\widetilde{c(\beta)}$ are not equivariant isotopic since they are even not homeomorphic.





Equivalence relation among orbit braids

How to define equivalence relation among orbit braids?

Isotopy with respect to the G-action

Let $\alpha, \beta : I \longrightarrow F_G(M, n)$ be two paths with the same endpoints. We say that $c(\alpha) \sim^G_{iso} c(\beta)$ (**isotopic with respect to the G-action** in $M \times I$) if there exist n homotopy maps $\widehat{h}_i : I \times I \longrightarrow M \times I$ given by $\widehat{h}_i(s, t) = (h_i(s, t), s)$, $i = 1, ..., n$, such that

(1) $\coprod_{i=1}^n \widehat{h}_i(s, 0) = c(\alpha)$ and $\coprod_{i=1}^n \widehat{h}_i(s, 1) = c(\beta)$;

(2) $\coprod_{i=1}^n \widehat{h}_i(0, t) = c(\alpha)|_{s=0} = c(\beta)|_{s=0}$ and $\coprod_{i=1}^n \widehat{h}_i(1, t) = c(\alpha)|_{s=1} = c(\beta)|_{s=1}$;

(3) For any $(s, t) \in I \times I$, if $i \neq j$ then $G(h_i(s, t)) \cap G(h_j(s, t)) = \emptyset$.

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(2) $\coprod_{i=1}^n \widehat{h}_i(0, t) = c(\alpha)|_{s=0} = c(\beta)|_{s=0}$ and $\coprod_{i=1}^n \widehat{h}_i(1, t) = c(\alpha)|_{s=1} = c(\beta)|_{s=1}$;

(3) For any $(s, t) \in I \times I$, if $i \neq j$ then $G(h_i(s, t)) \cap G(h_j(s, t)) = \emptyset$.

Proposition

Let $\alpha, \beta : I \longrightarrow F_G(M, n)$ be two paths with the same endpoints. Then

$$\alpha \simeq \beta (\text{rel } \partial I) \iff c(\alpha) \sim^G_{iso} c(\beta).$$

Equivalence relation among orbit braids

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We say that $\widetilde{c(\alpha)}$ and $\widetilde{c(\beta)}$ are **equivalent**, denoted by $\widetilde{c(\alpha)} \sim \widetilde{c(\beta)}$, if there are some g and h in $G^{\times n}$ such that $c(g\alpha) \sim_{iso}^{G} c(h\beta)$.

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In terms of homotopy

Proposition

$\widetilde{c(\alpha)} \sim \widetilde{c(\beta)} \Longleftrightarrow$ there are two paths α' and β' with $\widetilde{c(\alpha')} = \widetilde{c(\alpha)}$ and $\widetilde{c(\beta')} = \widetilde{c(\beta)}$, such that $\alpha' \simeq \beta'$ rel ∂I.

Orbit braid groups

- Let $\mathcal{B}_n^{orb}(M, G)$ be the set consisting of the equivalence classes of all orbit braids at orbit base point $\widetilde{c(\mathbf{x})}$ in $M \times I$.

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Remark: Each class $[\widetilde{c(\alpha)}]$ in $\mathcal{B}_n^{orb}(M,G)$ determines a unique pair $(g,\sigma) \in G^{\times n} \times \Sigma_n$. **Key Point!**

Subgroups of orbit braid groups

(1) Those classes $[\widetilde{c(\alpha)}]$ with $\alpha(1) \in G^{\times n}(\mathbf{x})$ of $\mathcal{B}_n^{orb}(M, G)$ form a subgroup of $\mathcal{B}_n^{orb}(M, G)$, which is called the **pure orbit braid group**, denoted by $\mathcal{P}_n^{orb}(M, G)$.

(2) Those classes $[\widetilde{c(\alpha)}]$ with $\alpha(1) \in \Sigma_n(\mathbf{x}) = \{\mathbf{x}_\sigma | \sigma \in \Sigma_n\}$ of $\mathcal{B}_n^{orb}(M, G)$ form a subgroup of $\mathcal{B}_n^{orb}(M, G)$, which is called the **braid group**, denoted by $\mathcal{B}_n(M, G)$.

(3) Those classes $[\widetilde{c(\alpha)}]$ with $\alpha(1) = \mathbf{x}$ of $\mathcal{B}_n^{orb}(M, G)$ form a subgroup of $\mathcal{B}_n^{orb}(M, G)$, which is called the **pure braid group**, denoted by $\mathcal{P}_n(M, G)$.

Homotopy description–Extended fundamental group

- Let $\pi_1^E(F_G(M,n),\mathbf{x},\mathbf{x}^{orb})$ be the set consisting of the homotopy classes relative to ∂I of all paths $\alpha : I \longrightarrow F_G(M,n)$ with $\alpha(0) = \mathbf{x}$ and $\alpha(1) \in \mathbf{x}^{orb}$, where $\mathbf{x}^{orb} = \{g\mathbf{x}_\sigma | g \in G^{\times n}, \sigma \in \Sigma_n\}$ is the orbit set at $\mathbf{x}$ under two actions of $G^{\times n}$ and Σ_n.

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- we can endow an operation $\bullet$ on $\pi_1^E(F_G(M,n),\mathbf{x},\mathbf{x}^{orb})$ defined by

$$[\alpha]\bullet[\beta]=[\alpha\circ(g\beta_\sigma)] \tag{1}$$

  where $(g,\sigma)\in G^{\times n}\times\Sigma_n$ is the unique pair determined by $[\widetilde{c(\alpha)}]$.

Homotopy description of orbit braid group

Theorem

$\pi_1^E(F_G(M,n),\mathbf{x},\mathbf{x}^{orb})$ forms a group under the operation $\bullet$. Furthermore, the map

$$\Lambda : \pi_1^E(F_G(M,n),\mathbf{x},\mathbf{x}^{orb}) \longrightarrow \mathcal{B}_n^{orb}(M,G)$$

given by $[\alpha] \longmapsto [\widetilde{c(\alpha})]$ is an isomorphism.

$\pi_1^E(F_G(M,n),\mathbf{x},\mathbf{x}^{orb})$ is called the **extended fundamental group** of $F_G(M,n)$ at $\mathbf{x}^{orb}$.

Homotopy description of subgroups

Corollary

(1) $\mathcal{P}_n^{orb}(M, G) \cong \pi_1^E(F_G(M, n), \mathbf{x}, G^{\times n}(\mathbf{x}))$;

(2) $\mathcal{B}_n(M, G) \cong \pi_1^E(F_G(M, n), \mathbf{x}, \Sigma_n(\mathbf{x}))$;

(3) $\mathcal{P}_n(M, G) \cong \pi_1^E(F_G(M, n), \mathbf{x}, \mathbf{x}) = \pi_1(F_G(M, n), \mathbf{x})$.

Homotopy description of subgroups

Corollary

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(3) $\mathcal{P}_n(M,G) \cong \pi_1^E(F_G(M,n), \mathbf{x}, \mathbf{x}) = \pi_1(F_G(M,n), \mathbf{x})$.

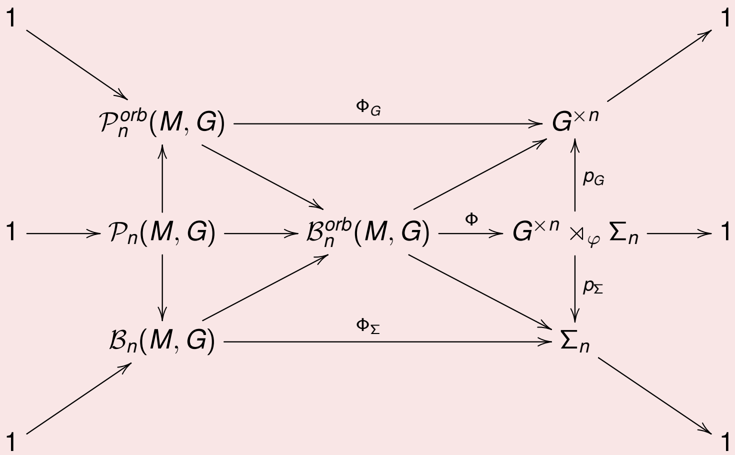
Remark: The above viewpoint can also be used in the theory of ordinary braids. Consider the case in which $G = \{e\}$. Then $\mathcal{B}_n^{orb}(M,G)$ degenerates into the ordinary braid group $B_n(M)$, which is isomorphic to the extended fundamental group $\pi_1^E(F(M,n), \mathbf{x}, \Sigma_n(\mathbf{x}))$ of $F(M,n)$ at $\Sigma_n(\mathbf{x})$. There is the following short exact sequence

$$1 \longrightarrow \pi_1(F(M,n), \mathbf{x}) \longrightarrow \pi_1^E(F(M,n), \mathbf{x}, \Sigma_n(\mathbf{x})) \longrightarrow \Sigma_n \longrightarrow 1$$

from which we see that $\pi_1^E(F(M,n), \mathbf{x}, \Sigma_n(\mathbf{x}))$ is actually the fundamental group of the unordered configuration space $F(M,n)/\Sigma_n$. However, the case of $G \neq \{e\}$ will be quite different.

Five short exact sequences

Theorem



Main point of proof

Let $\varphi : \Sigma_n \longrightarrow \mathrm{Aut}(G^{\times n})$ be a homomorphism defined by

$$\varphi(\sigma)(g) = g_\sigma = (g_{\sigma(1)}, ..., g_{\sigma(n)})$$

where $\sigma \in \Sigma_n$ and $g = (g_1, ..., g_n) \in G^{\times n}$. Then φ gives a semidirect product $G^{\times n} \rtimes_\varphi \Sigma_n$, where the operation $\cdot$ on $G^{\times n} \rtimes_\varphi \Sigma_n$ is given by

$$(g, \sigma) \cdot (h, \tau) = (gh_\sigma, \sigma\tau)$$

for $(g, \sigma), (h, \tau) \in G^{\times n} \rtimes_\varphi \Sigma_n$.

Main point of proof (continued)

Define a homomorphism

$$\Phi : \mathcal{B}_n^{orb}(M, G) \longrightarrow G^{\times n} \rtimes_{\varphi} \Sigma_n$$

by $\Phi([\widetilde{c(\alpha)}]) = (g, \sigma)$, where (g, σ) is the unique pair determined by $[\widetilde{c(\alpha)}]$.

Lemma

The homomorphism $\Phi : \mathcal{B}_n^{orb}(M, G) \longrightarrow G^{\times n} \rtimes_{\varphi} \Sigma_n$ is an epimorphism.

Two typical actions on $\mathbb{C}$

The geometric presentation of classical braid group $B_n(\mathbb{R}^2)$ in $\mathbb{R}^2 \times I$ gives us much more insights to the case of orbit braid group. Thus we begin with our work from the case of $\mathbb{C} \approx \mathbb{R}^2$ with the following two typical actions:

(I) $\mathbb{Z}_p \curvearrowright^{\phi_1} \mathbb{C}$ defined by $(e^{\frac{2k\pi \mathbf{i}}{p}}, z) \longmapsto e^{\frac{2k\pi \mathbf{i}}{p}} z$, which is non-free and fixes only the origin of $\mathbb{C}$, where p is a prime, and $\mathbb{Z}_p$ is regarded as the group $\{e^{\frac{2k\pi \mathbf{i}}{p}} | 0 \leq k < p\}$. If the action ϕ_1 is restricted to $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, then the action $\mathbb{Z}_p \curvearrowright^{\phi_1} \mathbb{C}^\times$ is free.

(II) $(\mathbb{Z}_2)^2 \curvearrowright^{\phi_2} \mathbb{C}$ defined by

$$\begin{cases} z \longmapsto \overline{z} \\ z \longmapsto -\overline{z}. \end{cases}$$

Orbit braid group $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}, \mathbb{Z}_p)$ of $F_{\mathbb{Z}_p}(\mathbb{C}, n)$

Proposition

$\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}, \mathbb{Z}_p)$ is generated by $\mathbf{b}_k$ $(1 \leq k \leq n-1)$ and $\mathbf{b}$, with relations

(1) $\mathbf{b}^p = e$;

(2) $(\mathbf{b}\mathbf{b}_1)^p = (\mathbf{b}_1\mathbf{b})^p$;

(3) $\mathbf{b}_k\mathbf{b} = \mathbf{b}\mathbf{b}_k \quad (k > 1)$;

(4) $\mathbf{b}_k\mathbf{b}_{k+1}\mathbf{b}_k = \mathbf{b}_{k+1}\mathbf{b}_k\mathbf{b}_{k+1}$;

(5) $\mathbf{b}_k\mathbf{b}_l = \mathbf{b}_l\mathbf{b}_k \quad (|k-l| > 1)$.

where $\mathbf{b}_k = [\widetilde{c(\alpha^{(k)})}]$ for $1 \leq k \leq n-1$ and $\mathbf{b} = [\widetilde{c(\beta)}]$ given by

$$\alpha^{(k)}(s) = (1+\mathbf{i}, \ldots, k+(k+1)\mathbf{i} + e^{-\frac{\pi}{2}\mathbf{i}(1-s)}, (k+1)+k\mathbf{i}+\mathbf{i}e^{\frac{\pi}{2}\mathbf{i}s}, \ldots, n+n\mathbf{i})$$

$$\beta(s) = ((1+\mathbf{i})e^{\frac{2\pi\mathbf{i}s}{p}}, 2+2\mathbf{i}, \ldots, n+n\mathbf{i}).$$

Orbit braid group $\mathcal{B}_n^{\text{orb}}(\mathbb{C}^\times, \mathbb{Z}_p)$ of $F_{\mathbb{Z}_p}(\mathbb{C}^\times, n)$

Proposition

$\mathcal{B}_n^{\text{orb}}(\mathbb{C}^\times, \mathbb{Z}_p)$ is genereated by $\mathbf{b_k}$ $(1 \leq k \leq n-1)$ and $\mathbf{b}'$, with relations:

(1) $(\mathbf{b}'\mathbf{b}_1)^p = (\mathbf{b}_1\mathbf{b}')^p$;

(2) $\mathbf{b}_k\mathbf{b}' = \mathbf{b}'\mathbf{b}_k \quad (k > 1)$;

(3) $\mathbf{b}_k\mathbf{b}_{k+1}\mathbf{b}_k = \mathbf{b}_{k+1}\mathbf{b}_k\mathbf{b}_{k+1}$;

(4) $\mathbf{b}_k\mathbf{b}_l = \mathbf{b}_l\mathbf{b}_k \quad (|k-l| > 1)$.

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$$\beta(s) = ((1+\mathbf{i})e^{\frac{2\pi\mathbf{i}s}{p}}, 2+2\mathbf{i}, \dots, n+n\mathbf{i}).$$

Orbit braid group $\mathcal{B}_n^{\text{orb}}(\mathbb{C}, \mathbb{Z}_2^2)$ of $F_{\mathbb{Z}_2^2}(\mathbb{C}, n)$

Proposition

$\mathcal{B}_n^{\text{orb}}(\mathbb{C}, \mathbb{Z}_2^2)$ is genereated by $\mathbf{b}_k$ $(1 \leq k \leq n-1)$, $\mathbf{b}^x$ and $\mathbf{b}^y$ with relations

(1) $(\mathbf{b}^x)^2 = (\mathbf{b}^y)^2 = e$;

(2) $\mathbf{b}^x\mathbf{b}^y = \mathbf{b}^y\mathbf{b}^x$;

(3) $\mathbf{b}^x\mathbf{b}_1\mathbf{b}^x\mathbf{b}_1 = \mathbf{b}_1\mathbf{b}^x\mathbf{b}_1\mathbf{b}^x, \quad \mathbf{b}^y\mathbf{b}_1\mathbf{b}^y\mathbf{b}_1 = \mathbf{b}_1\mathbf{b}^y\mathbf{b}_1\mathbf{b}^y$;

(4) $\mathbf{b}_k\mathbf{b}^x = \mathbf{b}^x\mathbf{b}_k, \quad \mathbf{b}_k\mathbf{b}^y = \mathbf{b}^y\mathbf{b}_k \quad (k > 1)$;

(5) $\mathbf{b}_k\mathbf{b}_{k+1}\mathbf{b}_k = \mathbf{b}_{k+1}\mathbf{b}_k\mathbf{b}_{k+1}$;

(6) $\mathbf{b}_k\mathbf{b}_l = \mathbf{b}_l\mathbf{b}_k \quad (|k-l| > 1)$.

Generators of orbit braid group $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}, \mathbb{Z}_2^2)$

(1) $\mathbf{b}_k$ is chosen as $[\widetilde{c(\alpha^{(k)})}]$ where

$$\alpha^{(k)}(s) = (1+\mathbf{i}, \dots, k+(k+1)\mathbf{i}+e^{-\frac{\pi}{2}\mathbf{i}(1-s)}, (k+1)+k\mathbf{i}+\mathbf{i}e^{\frac{\pi}{2}\mathbf{i}s}, \dots, n+n\mathbf{i});$$

(2) $\mathbf{b}^x$ is chosen as $[\widetilde{c(\alpha^x)}]$ where α^x is the path given by

$$\alpha^x(s) = (1+(1-2s)\mathbf{i}, 2+2\mathbf{i}, ..., n+n\mathbf{i})$$

such that α_1^x and $\overline{\alpha_1^x}$ intersect at x-axis$\times I$;

(3) $\mathbf{b}^y$ is chosen as $[\widetilde{c(\alpha^y)}]$ where α^y is the path given by

$$\alpha^y(s) = ((1-2s)+\mathbf{i}, 2+2\mathbf{i}, ..., n+n\mathbf{i})$$

such that α_1^y and $-\overline{\alpha_1^y}$ intersect at y-axis$\times I$.

Relation with generalized braid group

- It is known from Goryunov's work: two orbit configuration spaces $F_{\mathbb{Z}_2}(\mathbb{C}, n)$ and $F_{\mathbb{Z}_2}(\mathbb{C}^\times, n)$ are classifying spaces of two generalised pure braid groups $P(D_n)$ and $P(B_n)$.
- In the viewpoint of Brieskorn, $F_{\mathbb{Z}_2}(\mathbb{C}, n) = Y_{D_n}$ so

$$1 \longrightarrow P(D_n) \longrightarrow Br(D_n) \longrightarrow D_n \longrightarrow 1$$

and $F_{\mathbb{Z}_2}(\mathbb{C}^\times, n) = Y_{B_n}$ so

$$1 \longrightarrow P(B_n) \longrightarrow Br(B_n) \longrightarrow B_n \longrightarrow 1$$

- In our viewpoint, there are

$$1 \longrightarrow \mathcal{P}_n(\mathbb{C}, \mathbb{Z}_2) \longrightarrow \mathcal{B}_n^{orb}(\mathbb{C}, \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2^n \rtimes_\varphi \Sigma_n \longrightarrow 1$$

$$1 \longrightarrow \mathcal{P}_n(\mathbb{C}^\times, \mathbb{Z}_2) \longrightarrow \mathcal{B}_n^{orb}(\mathbb{C}^\times, \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2^n \rtimes_\varphi \Sigma_n \longrightarrow 1$$

It can be checked that $\mathbb{Z}_2^n \rtimes_\varphi \Sigma_n \cong B_n$

Relation with generalized braid group

- For the case of $F_{\mathbb{Z}_2}(\mathbb{C}^\times, n)$, two viewpoints are identical. In this case, $F_{\mathbb{Z}_2}(\mathbb{C}^\times, n) = Y_{B_n}$, so that

$$\mathbf{Br(B_n)} \cong \mathcal{B}_{\mathbf{n}}^{\mathbf{orb}}(\mathbb{C}^\times, \mathbb{Z}_\mathbf{2})$$

- For the case of $F_{\mathbb{Z}_2}(\mathbb{C}, n)$, two viewpoints are not the same. However,

  $Br(D_n)$ is isomorphic to a subgroup of $\mathcal{B}_n^{\mathbf{orb}}(\mathbb{C}, \mathbb{Z}_2)$.

Thank You!