Newton-Okounkov polytopes of flag varieties from cluster algebras

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Toric degenerations

Let Z be an irreducible normal projective variety over $\mathbb{C}$, and $\mathcal{L}$ an ample line bundle on Z.

Definition

A **toric degeneration** of $(Z, \mathcal{L})$ is a flat morphism

$$\pi\colon \mathfrak{X} = \mathrm{Proj}(R) \to \mathbb{C}$$

such that $(\pi^{-1}(t), \mathcal{O}_{\mathfrak{X}}(1)|_{\pi^{-1}(t)}) \simeq (Z, \mathcal{L})$ for all $t \in \mathbb{C}^\times$, and $\pi^{-1}(0)$ is an irreducible normal projective toric variety.

Aim

to understand relations between the following two constructions of toric degenerations:

- Newton-Okounkov bodies (Anderson 2013),
- cluster algebras (Gross-Hacking-Keel-Kontsevich 2018).

Newton-Okounkov bodies

- Z: an irreducible normal projective variety over $\mathbb{C}$ ($m := \dim_{\mathbb{C}}(Z)$),
- $\mathcal{L}$: an ample line bundle on Z,
- $\tau \in H^0(Z, \mathcal{L})$: a nonzero section.

Assume that Z is rational, and fix an identification

$$\mathbb{C}(Z) \simeq \mathbb{C}(t_1, \ldots, t_m).$$

We take a total order $<$ on $\mathbb{Z}^m$, respecting the addition, which induces a total order $<$ on the set of Laurent monomials in $t_1, \ldots, t_m$. Then, the **lowest term valuation** $v \colon \mathbb{C}(Z) \setminus \{0\} \to \mathbb{Z}^m$ is defined as follows:

$$v(f/g) := v(f) - v(g), \text{ and}$$
$$v(f) := (a_1, \ldots, a_m) \Leftrightarrow f = ct_1^{a_1} \cdots t_m^{a_m} + (\text{higher terms w.r.t. } <)$$

for $f, g \in \mathbb{C}[t_1, \ldots, t_m] \setminus \{0\}$, where $c \in \mathbb{C}^\times$.

Newton-Okounkov bodies

We define a semigroup $S(Z, \mathcal{L}, v, \tau) \subset \mathbb{Z}_{>0} \times \mathbb{Z}^m$, a real closed convex cone $C(Z, \mathcal{L}, v, \tau) \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^m$ and a convex set $\Delta(Z, \mathcal{L}, v, \tau) \subset \mathbb{R}^m$ by

$$S(Z, \mathcal{L}, v, \tau) \coloneqq \{(k, v(\sigma/\tau^k)) \mid k \in \mathbb{Z}_{>0},\ \sigma \in H^0(Z, \mathcal{L}^{\otimes k}) \setminus \{0\}\},$$

$C(Z, \mathcal{L}, v, \tau)$: the smallest real closed cone containing $S(Z, \mathcal{L}, v, \tau)$,

$$\Delta(Z, \mathcal{L}, v, \tau) \coloneqq \{\mathbf{a} \in \mathbb{R}^m \mid (1, \mathbf{a}) \in C(Z, \mathcal{L}, v, \tau)\}.$$

Definition (Lazarsfeld-Mustata 2009, Kaveh-Khovanskii 2012)

The convex set $\Delta(Z, \mathcal{L}, v, \tau)$ is called a **Newton-Okounkov body** of Z.

Theorem (Anderson 2013)

If the semigroup $S(Z, \mathcal{L}, v, \tau)$ is finitely generated and saturated, then there exists a toric degeneration of $(Z, \mathcal{L})$ to the normal projective toric variety corresponding to $\Delta(Z, \mathcal{L}, v, \tau)$.

Newton-Okounkov bodies of flag varieties

- G: a connected, simply-connected semisimple algebraic group over $\mathbb{C}$,
- $B \subset G$: a Borel subgroup,
- P_+: the set of dominant integral weights.

Definition

The quotient variety G/B is called the **full flag variety**.

Proposition

There exists a natural bijective map

$$P_+ \xrightarrow{\sim} \{\text{line bundles on } G/B \text{ generated by global sections}\},$$
$$\lambda \mapsto \mathcal{L}_\lambda.$$

Newton-Okounkov bodies of flag varieties

The Newton-Okounkov bodies of $(G/B, \mathcal{L}_\lambda)$ realize the following representation-theoretic polytopes:

- Berenstein-Littelmann-Zelevinsky's string polytopes (Kaveh 2015),
- Nakashima-Zelevinsky polytopes (F.-Naito 2017),
- Feigin-Fourier-Littelmann-Vinberg polytopes (Feigin-Fourier-Littelmann 2017, Kiritchenko 2017).

Question

How are these polytopes related?

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Cluster varieties

Let

$$\mathcal{A} = \bigcup_t \mathcal{A}_t = \bigcup_t \mathrm{Spec}(\mathbb{C}[\underbrace{x_1(t)^{\pm 1}, \ldots, x_n(t)^{\pm 1}}_{\text{unfrozen}}, \underbrace{x_{n+1}(t)^{\pm 1}, \ldots, x_m(t)^{\pm 1}}_{\text{frozen}}])$$

be a cluster variety (Fock-Goncharov 2009), where t runs over the set of seeds which are mutually mutation equivalent, and the tori are glued via the following birational cluster mutations:

$$\mu_k^*(x_i(t')) = \begin{cases} x_i(t) & (i \neq k), \\ x_k(t)^{-1}\left(\prod_{b_{k,j}^{(t)}>0} x_j(t)^{b_{k,j}^{(t)}} + \prod_{b_{k,j}^{(t)}<0} x_j(t)^{-b_{k,j}^{(t)}}\right) & (i = k) \end{cases}$$

if $t' = \mu_k(t)$, where $B(t) = (b_{k,j}^{(t)})_{k,j}$ is the exchange matrix of t.

Definition (Berenstein-Fomin-Zelevinsky 2005)

The ring $\mathbb{C}[\mathcal{A}]$ of regular functions is called an **upper cluster algebra**.

Cluster varieties

Denoting the dual torus of $\mathcal{A}_t$ by $\mathcal{A}_t^\vee$, we have

$$\mathcal{A} = \bigcup_t \mathcal{A}_t \quad \xleftarrow{\text{"mirror"}} \!\!\! \longrightarrow \mathcal{A}^\vee = \bigcup_t \mathcal{A}_t^\vee.$$

The space $\mathcal{A}^\vee$ is called the **Fock-Goncharov dual**, and defined to be the Langlands dual of the $\mathcal{X}$-cluster variety. Since the gluing maps of $\mathcal{A}^\vee$ are given by subtraction-free rational functions, we obtain the set $\mathcal{A}^\vee(\mathbb{R}^T)$ of $\mathbb{R}^T$-valued points, where $\mathbb{R}^T$ is a semifield $(\mathbb{R}, \max, +)$. More precisely, $\mathcal{A}^\vee(\mathbb{R}^T)$ is defined by gluing $\mathcal{A}_t^\vee(\mathbb{R}^T) = \mathbb{R}^m$ via the following tropicalized cluster mutation:

$$\mu_k^T \colon \mathcal{A}_t^\vee(\mathbb{R}^T) \to \mathcal{A}_{t'}^\vee(\mathbb{R}^T),\ (g_1, \ldots, g_m) \mapsto (g_1', \ldots, g_m'),$$

where

$$g_i' = \begin{cases} g_i + \max\{b_{k,i}^{(t)}, 0\} g_k - b_{k,i}^{(t)} \min\{g_k, 0\} & (i \neq k), \\ -g_k & (i = k). \end{cases}$$

Gross-Hacking-Keel-Kontsevich's toric degenerations

For each t, we have

$$\mathcal{A}^\vee(\mathbb{R}^T) \simeq \mathcal{A}_t^\vee(\mathbb{R}^T) = \mathbb{R}^m.$$

We fix $\Xi \subset \mathcal{A}^\vee(\mathbb{R}^T)$ such that its image in $\mathcal{A}_t^\vee(\mathbb{R}^T) = \mathbb{R}^m$, denoted by Ξ_t, is an m-dimensional rational convex polytope for each t.

Theorem (Gross-Hacking-Keel-Kontsevich 2018)

Under some conditions on $\mathcal{A}$ and Ξ, there exists a $\mathbb{Z}_{\geq 0}$-graded ring R_t for each t together with a flat morphism $\pi_t \colon \mathrm{Proj}(R_t) \to \mathbb{C}^m$ such that

- *$\pi_t^{-1}(z)$ for $z \in (\mathbb{C}^\times)^m$ is a normal projective variety which compactifies $\mathcal{A}$,*
- *$\pi_t^{-1}(0)$ is the normal projective toric variety corresponding to $\Xi_t \subset \mathbb{R}^m$.*

extended g-vectors

Theorem (Fomin-Zelevinsky 2007, Derksen-Weyman-Zelevinsky 2010, Gross-Hacking-Keel-Kontsevich 2018)

For all t, t' and $1 \leq i \leq m$,

$$x_i(t') \in x_1(t)^{g_1} \cdots x_m(t)^{g_m} (1 + \sum_{0 \neq (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n} \mathbb{Z} \hat{y}_1^{a_1} \cdots \hat{y}_n^{a_n})$$

for some $g_t(x_i(t')) := (g_1, \ldots, g_m) \in \mathbb{Z}^m$, where

$$\hat{y}_i := x_1(t)^{b_{i,1}^{(t)}} \cdots x_m(t)^{b_{i,m}^{(t)}}.$$

The vector $g_t(x_i(t')) \in \mathbb{Z}^m$ is called the **extended g-vector** of $x_i(t')$.

g-vectors as higher rank valuations

Definition (Qin 2017)

For each t, define a partial order $\leq_t$ on $\mathbb{Z}^m$ by

$$g' \leq_t g \Leftrightarrow g - g' \in \sum_{1 \leq i \leq n} \mathbb{Z}_{\geq 0}(b_{i,1}^{(t)}, \ldots, b_{i,m}^{(t)}).$$

This $\leq_t$ is called the **dominance order** associated with t.

Since $\mathcal{A}_t$ is an open subvariety of $\mathcal{A}$, we have

$$\mathbb{C}(\mathcal{A}) = \mathbb{C}(\mathcal{A}_t) = \mathbb{C}(x_1(t), \ldots, x_m(t)).$$

We fix a total order $\leq_t$ on $\mathbb{Z}^m$ which refines the dominance order $\leq_t$.

Definition

For each t, define a valuation $v_t \colon \mathbb{C}(\mathcal{A}) \setminus \{0\} \to \mathbb{Z}^m$ to be the lowest term valuation associated with $\leq_t$.

g-vectors as higher rank valuations

Proposition

For all t, t' and $1 \leq i \leq m$, the equality $v_t(x_i(t')) = g_t(x_i(t'))$ holds.

Proposition

The valuation v_t generalizes the extended g-vectors of the theta function basis.

Corollary

Under Gross-Hacking-Keel-Kontsevich's toric degeneration, the polytope Ξ_t is identical to the Newton-Okounkov body of $\pi_t^{-1}(z)$ associated with v_t.

Let Z be an irreducible normal projective variety which is birational to $\mathcal{A}$.

Problem

Compute Newton-Okounkov bodies of Z associated with v_t.

Cluster structures

Let U^- be the unipotent radical of the opposite Borel subgroup B^-, and

$$G/B = \bigsqcup_{w \in W} BwB/B$$

the Bruhat decomposition of G/B, where W is the Weyl group.

Definition

For $w \in W$, the **unipotent cell** $U_w^- \subset G$ is defined by

$$U_w^- := BwB \cap U^-.$$

Theorem (Berenstein-Fomin-Zelevinsky 2005)

The coordinate ring $\mathbb{C}[U_w^-]$ admits an upper cluster algebra structure.

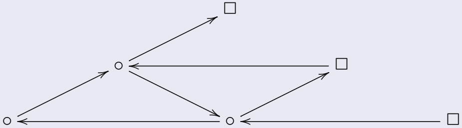
There exists $w_0 \in W$, called the **longest element**, such that the natural projection $G \twoheadrightarrow G/B$ induces an open embedding $U_{w_0}^- \hookrightarrow G/B$.

Relation with string polytopes

Each reduced word $\mathbf{i} = (i_1, \ldots, i_m)$ for w_0 induces a seed $t_{\mathbf{i}}$ for $U^-_{w_0}$.

Example

Let $G = SL_4(\mathbb{C})$, and $\mathbf{i} = (1, 2, 1, 3, 2, 1)$. Then, the exchange matrix $B(t_{\mathbf{i}}) = (b^{(t_{\mathbf{i}})}_{i,j})_{i,j}$ corresponds to the following quiver:



For instance, $b^{(t_{\mathbf{i}})}_{i,j} = 1$ and $b^{(t_{\mathbf{i}})}_{j,i} = -1$ (if defined and) if there exists $i \to j$.

Theorem (F.-Oya)

The Newton-Okounkov body $\Delta(G/B, \mathcal{L}_\lambda, v_{t_{\mathbf{i}}}, \tau)$ is unimodularly equivalent to the string polytope $\Delta_{\mathbf{i}}(\lambda)$ associated with $\mathbf{i}$ and λ.

Relation with Nakashima-Zelevinsky polytopes

Berenstein-Fomin-Zelevinsky (1996) and Berenstein-Zelevinsky (1997) introduced the **twist automorphism** $\eta_w \colon U_w^- \xrightarrow{\sim} U_w^-$ to solve a factorization problem, called the Chamber Ansatz. It is defined by

$$\eta_w(u) := [u^T \overline{w}]_-,$$

where u^T is the transpose of u, and $\overline{w} \in G$ is a lift for w. In addition, $[u^T \overline{w}]_-$ is the U^--part of $u^T \overline{w}$ under the Gaussian decomposition. Then, there exists a sequence $\overrightarrow{\mu} = \mu_{k_1}\mu_{k_2}\cdots\mu_{k_l}$ of mutations which corresponds to η_w in some sense (Geiss-Leclerc-Schröer 2011, 2012, ...). We define a seed $\hat{t}_{\mathbf{i}}$ by

$$\hat{t}_{\mathbf{i}} := \overrightarrow{\mu}^{-1}(t_{\mathbf{i}}).$$

Theorem (F.-Oya)

The Newton-Okounkov body $\Delta(G/B, \mathcal{L}_\lambda, v_{\hat{t}_{\mathbf{i}}}, \tau)$ is unimodularly equivalent to the Nakashima-Zelevinsky polytope $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ associated with $\mathbf{i}$ and λ.

Simply-laced case

In simply-laced case, using the monoidal categorification of the cluster algebra by Kang-Kashiwara-Kim-Oh (2018), we obtain the following.

Theorem (F.-Oya)

If G is of simply-laced, then the following hold.

(1) For each t, the Newton-Okounkov body $\Delta(G/B, \mathcal{L}_\lambda, v_t, \tau)$ is a rational convex polytope which gives a toric degeneration of G/B.

(2) If $t' = \mu_k(t)$, then the following equality holds:

$$\Delta(G/B, \mathcal{L}_\lambda, v_{t'}, \tau) = \mu_k^T(\Delta(G/B, \mathcal{L}_\lambda, v_t, \tau)).$$

Corollary

If G is of simply-laced, then the string polytopes $\Delta_{\mathbf{i}}(\lambda)$ and the Nakashima-Zelevinsky polytopes $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ are all related by tropicalized cluster mutations.