Generalized moment-angle complexes and collectively unavoidable complexes

Rade T. Živaljević
Mathematical Institute SASA, Belgrade

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Let (X, A) be a pair of spaces and let K be an abstract simplicial complex, $K \subseteq 2^{[m]} \quad (m = N + 1)$.

The associated *Generalized Moment-Angle Complex* (K-power) is the space,

$$\mathcal{Z}_K(X, A) = \operatorname{colim}_{\sigma \in K}(X, A)^\sigma = \operatorname{colim}_{\sigma \in K}\left(\prod_{i \in \sigma} X \times \prod_{j \notin \sigma} A\right) \subseteq X^m.$$

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For $x = (x_i) \in X^m$ let $I_A(x) := \{i \in [m] \mid x_i \notin A\}$.

Then

$$\mathcal{Z}_K(X, A) = \{x \in X^m \mid I_A(x) \in K\}.$$

Alexander dual pairs $\langle K, K^\circ \rangle$

$$K * L = \{A \uplus B \mid A \in K, B \in L\}\,.$$

$$K *_\Delta L = \{A \uplus B \mid A \in K, B \in L \text{ and } A \cap B = \emptyset\}\,.$$

$$K^\circ = \{A \subset [m] \mid A^c \notin K\} \qquad \text{is the Alexander dual of } K\,.$$

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Proposition: (V. Welker, V. Grujić)

$$\mathcal{Z}_K(X, A) \uplus \mathcal{Z}_{K^\circ}(X, A^c) = X^m .$$

Proof: For each $x \in X^m$ either $I_A(x) \in K$ or $I_{A^c}(x) \in K^\circ$, but not both! Indeed, $I_A(x) \cap I_{A^c}(x) = \emptyset$.

Collectively unavoidable complexes

Definition: An ordered r-tuple $\mathcal{K} = \langle K_1, \ldots, K_r \rangle$ of subcomplexes of $2^{[m]}$ is *collectively r-unavoidable* if for each **ordered collection** $(A_1, \ldots, A_r)$ of disjoint sets in $[m]$ there exists i such that $A_i \in K_i$.

Example: The pair $\langle K, K^\circ \rangle$ is collectively unavoidable. Indeed, if (A_1, A_2) is an ordered pair of disjoint sets then either $A_1 \in K$ or (in the opposite case) $A_1^c \in K^\circ$, which implies $A_2 \in K^\circ$.

A complex $K \subseteq 2^{[r]}$ is by definition *r-unavoidable* if the r-tuple $\langle K, K, \ldots, K \rangle$ is collectively r-unavoidable.

Collectively unavoidable complexes and moment-angle complexes

Collectively unavoidable families $\mathcal{K} = \langle K_i \rangle_{i=1}^r$ admit a characterization in the language of generalized moment-angle complexes.

Proposition: Let X be a topological space and $\{A_i\}_{i=1}^r$ a family of its subspaces which are *complementary* in the sense that $X = A_i \cup A_j$ for each $i \neq j$. Then if $\mathcal{K} = \langle K_i \rangle_{i=1}^r = \langle K_1, \dots, K_r \rangle$ is a collectively r-unavoidable family of subcomplexes of the N-dimensional simplex $\Delta_N = 2^{[N+1]}$ then

$$X^{N+1} = \mathcal{Z}_{K_1}(X, A_1) \cup \dots \cup \mathcal{Z}_{K_r}(X, A_r)\,. \qquad (1)$$

Conversely, if (1) holds for each X and each family $\{A_i\}_{i=1}^r$ of complementary subspaces in X then $\mathcal{K} = \langle K_i \rangle_{i=1}^r$ is a collectively r-unavoidable family of simplicial complexes.

Proof of the Proposition

It follows from the definition that

$$\mathcal{Z}_{K_i}(X, A_i) = \{x \in X^{N+1} \mid I_i(x) \in K_i\}$$

where $I_i(x) := \{j \in [N+1] \mid x_j \notin A_i\}$.
$A_i \cup A_j = X$ for each $i \neq j$ implies $I_i(x) \cap I_j(x) = \emptyset$. By collective unavoidability of $\mathcal{K} = \langle K_i \rangle_{i=1}^r$, for each $x \in X^{N+1}$ there exists $i \in [r]$ such that $\{I_i(x) \in K_i\}$, and the relation (1) is an immediate consequence.

Conversely, assume that $\mathcal{K} = \langle K_i \rangle_{i=1}^r$ is not collectively unavoidable. By definition there exist pairwise disjoint subsets $\{I_j\}_{j=1}^r$ of $[N+1]$ such that $I_i \notin K_i$ for each $i \in [r]$. Let $X = [N+1]$ and let $A_i := [N+1] \setminus I_i$. Let $x : [N+1] \to X$ be the identity map, ($x_i = i$ for each $i \in [N+1]$). Then,

$$x \in X^{N+1} \setminus \bigcup_{i=1}^{r} \mathcal{Z}_{K_i}(X, A_i)\,.$$

A canonical family of complementary sets

Let $W = \bigvee_{j=1}^{m} I_j = \bigvee_{j=1}^{m}[0,1]$ be the Kowalski *m-hedgehog space* obtained by gluing m "spikes" along 0. Let W_i are its $(m-1)$-hedgehog subspaces obtained by removing the spike I_i.

Then $\{W_i\}_{i=1}^{m}$ is a family of complementary set and if $\mathcal{K} = \langle K_i \rangle_{i=1}^{r} = \langle K_1, \dots, K_r \rangle$ is a collectively r-unavoidable family of complexes then

$$W^{N+1} = \mathcal{Z}_{K_1}(W, W_1) \cup \cdots \cup \mathcal{Z}_{K_r}(W, W_r)\,. \qquad (2)$$

Van Kampen-Flores type theorem for collectively unavoidable complexes

Theorem A. $\mathcal{K} = \langle K_i \rangle_{i=1}^r = \langle K_1, \dots, K_r \rangle$ is a *collectively r-unavoidable* family of subcomplexes of the N-dimensional simplex $\Delta_N = 2^{[N+1]}$, where $r = p^\nu$ is a power of a prime.

Assume that there exists $k \geq 1$ such that for each i

$$\Delta_N^{(k-1)} \subseteq K_i \subseteq \Delta_N^{(k)}$$

where $\Delta_N^{(k)}$ is the k-dimensional skeleton of Δ_N.

Suppose that $N \geq (r-1)(d+2)$.

Theorem A conclusion

Then for each continuous map $f : \Delta_N \to \mathbb{R}^d$, there exist vertex-disjoint faces $\sigma_1, \ldots, \sigma_r$ of Δ_N such that

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$$

and

$$\sigma_1 \in K_1, \sigma_2 \in K_2, \ldots, \sigma_r \in K_r \, .$$

[JPZ-1] D. Jojić, G. Panina, R. Živaljević, *A Tverberg type theorem for collectively unavoidable complexes*, *Israel J. Math.*

Van Kampen-Flores theorem

Theorem: (Van Kampen-Flores 1930s) One can always find two intersecting triangles in each collection of 7 points in four-dimensional euclidean space.

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More generally, for each collection $C \subset \mathbb{R}^{2d}$ of cardinality $(2d+3)$ there exist two disjoint sub-collections C_1 and C_2 of size $\leq (d+1)$ such that,

$$\mathrm{conv}(C_1) \cap \mathrm{conv}(C_2) \neq \emptyset.$$

Van Kampen-Flores theorem non-linear version

Theorem: For each continuous map,

$$f : \Delta_N \to \mathbb{R}^{2d}$$

where $N = 2d + 2$ and Δ_N is an N-dimensional simplex, there exist two disjoint faces σ_1 and σ_2 of Δ_N such that $\dim(\sigma_i) \leq d$ and

$$f(\sigma_1) \cap f(\sigma_2) \neq \emptyset.$$

Balanced generalized van Kampen-Flores theorem

Theorem B: Let $r \geq 2$ be a prime power, $d \geq 1$, $N \geq (r-1)(d+2)$, and $rk + s \geq (r-1)d$ for integers $k \geq 0$ and $0 \leq s < r$. Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$, there are r pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of Δ_N such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$, with $\dim \sigma_i \leq k+1$ for $1 \leq i \leq s$ and $\dim \sigma_i \leq k$ for $s < i \leq r$.

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D. Jojić, S.T. Vrećica, R.T. Živaljević.
Symmetric multiple chessboard complexes and a new theorem of Tverberg type, J. Algebraic Combin., 46 (2017), 15–31.

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The theorem confirms a conjecture in [BFZ14] Blagojević, Frick, and Ziegler (Conjecture 6.6 in, Tverberg plus constraints, Bull. London Math. Soc., 46 (2014).)

Consequences

(1) Implies positive answer to the 'balanced case' of the problem whether each *admissible* r-tuple is *Tverberg prescribable*, ([BFZ14], Question 6.9];

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(5) The generalized van Kampen-Flores theorem ([BFZ14], Theorem 6.3), which improves upon earlier results of Sarkaria and Volovikov, follows for $s = 0$ and $k = \lceil \frac{r-1}{r} d \rceil$.

Counterexamples

Theorems A and B are **positive results** obtained after the fundamental progress leading to counterexamples to the continuous Tverberg-Van Kampen-Flores theorems in the non-prime power with central contributions by:

- I. Mabillard and U. Wagner;
- P.V.M. Blagojević, F. Frick, G. Ziegler;
- M. Özaydin;
- M. Gromov.

A. Skopenkov, A user's guide to the topological Tverberg Conjecture, Russian Math. Surveys, 73:2 (2018), 323–353. Earlier version: arXiv:1605.05141v4.

For the improved counterexamples see also

- S. Avvakumov, R. Karasev and A. Skopenkov, (2019).

Proofs of Theorems A and B

A central role is played by high connectivity results as illustrated by:

Theorem C: Suppose that $\mathcal{K} = \langle K_i \rangle_{i=1}^r = \langle K_1, \dots, K_r \rangle$ is a *collectively r-unavoidable* family of subcomplexes of $2^{[m]}$. Then the associated deleted join

$$DelJoin(\mathcal{K}) = K_1 *_\Delta K_2 *_\Delta \cdots *_\Delta K_r$$

is $(m - r - 1)$-connected.

D. Jojić, I. Nekrasov, G. Panina, R. Živaljević, *Alexander r-tuples and Bier complexes*, Publ. Inst. Math. (Beograd) (N.S.) 104(118) (2018), 1–22.

Connection with moment-angle complexes

Theorem:

$$Bier(K) := K *_\Delta K^\circ \simeq \check{\mathcal{Z}}_K(X; A) \cap \check{\mathcal{Z}}_{K^\circ}(X; B)\,.$$

where $X = [0, 1], A = [0, 1/2], B = [1/2, 1]$ and $\check{\mathcal{Z}}_K(X, A) := \mathcal{Z}_K(X, A) \setminus \{1/2\}^m$ is the "reduced" moment-angle complex.

Connection with moment-angle complexes

More generally

Theorem:

$$K_1 *_\Delta \cdots *_\Delta K_r \simeq \check{\mathcal{Z}}_{K_1}(W; W_1) \cap \cdots \cap \check{\mathcal{Z}}_{K_r}(W; W_r)$$

where $W = \bigvee_{i=1}^{m}[0,1]$ is the Kowalski *m-hedgehog space* and W_i are its $(m-1)$-hedgehog subspaces. The reduced moment-angle complex is obtained by removing the point $(0, 0, \ldots, 0)$.

Theorem C revisited

Theorem: Assume that $\{K_i\}_{i=1}^r$ is a family of subcomplexes of $2^{[m]} \cong \Delta_N$ such that

$$W^m = \mathcal{Z}_{K_1}(W, W_1) \cup \cdots \cup \mathcal{Z}_{K_r}(W, W_r)\,. \qquad (3)$$

Then the space

$$\check{\mathcal{Z}}_{K_1}(W; W_1) \cap \cdots \cap \check{\mathcal{Z}}_{K_r}(W; W_r)$$

is $(m - r - 1)$-connected.

D. Jojić, I. Nekrasov, G. Panina, R. Živaljević, Alexander r-tuples and Bier complexes, *Publ. Inst. Math. (Beograd) (N.S.)* 104(118) (2018), 1–22.

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