Two enriched poset polytopes

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joint work with Hidefumi Ohsugi (Kwansei Gakuin University)

This talk is based on

- H. Ohsugi and A. Tsuchiya, Enriched chain polytopes, *Israel J. Math.*, to appear.
- H. Ohsugi and A. Tsuchiya, Enriched order polytopes and enriched Hibi rings, arXiv:1906.04719.

1. Two poset polytopes

2. Two enriched poset polytopes

3. Gal Conjecture

This slide was uploaded in my researchmap page. If you need, please google “researchmap Akiyoshi Tsuchiya”.

Ehrhart polynomial

$\mathcal{P} \subset \mathbb{R}^d$: a lattice polytope of dimension d
(i.e., a convex polytope all of whose vertices are in $\mathbb{Z}^d$)
$m\mathcal{P} = \{m\mathbf{x} : \mathbf{x} \in \mathcal{P}\}$: the mth dilated polytope of $\mathcal{P}$
$L_{\mathcal{P}}(m) := |m\mathcal{P} \cap \mathbb{Z}^d|$: the Ehrhart polynomial of $\mathcal{P}$

Theorem (Ehrhart)

$L_{\mathcal{P}}(m)$ is a polynomial in m of degree d.

Remark

- *The constant term of $L_{\mathcal{P}}(m)$ is equal to 1;*
- *The leading coefficient of $L_{\mathcal{P}}(m)$ is equal to the volume of $\mathcal{P}$;*
- *The second leading coefficient of $L_{\mathcal{P}}(m)$ is equal to the half of the relative volume of the boundary of $\mathcal{P}$;*
- $|\mathrm{int}(m\mathcal{P}) \cap \mathbb{Z}^d| = (-1)^d L_{\mathcal{P}}(-m)$.

Two poset polytopes

$(P, <_P)$: a poset on $[d] := \{1, \ldots, d\}$.

Definition (Stanley)

The order polytope of P is

$$\mathcal{O}_P := \{\mathbf{x} \in [0,1]^d : x_i \leq x_j \text{ if } i <_P j\}.$$

The chain polytope of P is

$$\mathcal{C}_P := \{\mathbf{x} \in [0,1]^d : x_{i_1} + \cdots + x_{i_r} \leq 1 \text{ if } i_1 <_P \cdots <_P i_r \text{ is a chain in } P\}.$$

Proposition (Stanley)

$\mathcal{O}_P$ and $\mathcal{C}_P$ are lattice polytopes of dimension d.

Remark

$\mathcal{O}_P$ and $\mathcal{O}_{\overline{P}}$ are always isomorphic, where $\overline{P}$ is the dual poset of P.

Vertices of $\mathcal{O}_P$ and $\mathcal{C}_P$

$(P, <_P)$: a poset on $[d]$.

- $F \subset [d]$ is a filter of P if for any $x \in F$ and $y \in P$, it follows that $x <_P y \Rightarrow y \in F$.
- $A \subset [d]$ is an antichain of P if for any $x, y \in A$ with $x \neq y$, x and y are incomparable.

$\mathcal{F}(P)$: the set of filters of P.
$\mathcal{A}(P)$: the set of antichains of P.

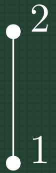
- For $X \subset [d]$, set $\mathbf{e}_X := \sum_{i \in X} \mathbf{e}_i$, where $\mathbf{e}_1, \dots, \mathbf{e}_d$ are the standard basis of $\mathbb{R}^d$. In particular, $\mathbf{e}_\emptyset = \mathbf{0}$.

Theorem (Stanley)

The set of vertices of $\mathcal{O}_P$ is $\{\mathbf{e}_F : F \in \mathcal{F}(P)\}$.
The set of vertices of $\mathcal{C}_P$ is $\{\mathbf{e}_A : A \in \mathcal{A}(P)\}$.

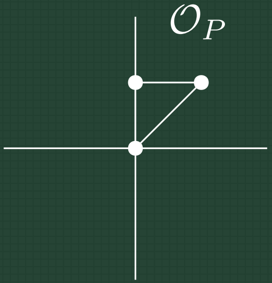
Example

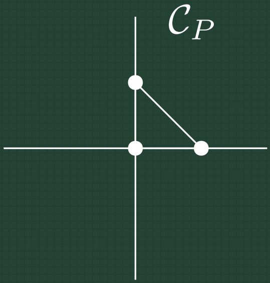
$P =$



$$\mathcal{F}(P) = \{\emptyset, \{2\}, \{1,2\}\}$$

$$\mathcal{A}(P) = \{\emptyset, \{1\}, \{2\}\}$$





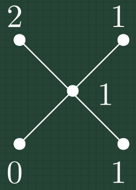
P-partition

$(P, <_P)$: a naturally labeled poset on $[d]$, i.e., $i <_P j \Rightarrow i < j$.

Definition

A map $f : P \to \mathbb{Z}_{\geq 0}$ is a P-partition if for any $i <_P j$,

$$f(i) \leq f(j).$$

$P =$ 

$(P, <_P)$: a naturally labeled poset on $[d]$.

Theorem (Stanley)

$$L_{\mathcal{O}_P}(m) = L_{\mathcal{C}_P}(m) = |\{f : P\text{-partitions with } f(i) \leq m\}|.$$

$\Omega_P(m) := |\{f : P\text{-partitions with } 1 \leq f(i) \leq m\}|$
: the order polynomial of P

G_P : the comparability graph of P, i.e., a simple graph on $[d]$ such that $\{x, y\}$ is an edge if and only if $x <_P y$ or $x >_P y$.

Corollary

$\Omega_P(m)$ depends only on G_P.

Enriched P-partition

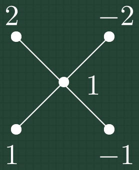
$(P, <_P)$: a naturally labeled poset on $[d]$.

Definition (Stembridge)

A map $f : P \to \mathbb{Z} \setminus \{0\}$ is an enriched P-parition if for any $i <_P j$,

- $|f(i)| \leq |f(j)|$;
- $|f(i)| = |f(j)| \Rightarrow f(j) > 0$.

$P =$



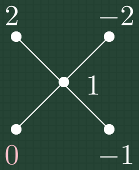
Left enriched P-partition

$(P, <_P)$: a naturally labeled poset on $[d]$.

Definition (Petersen)

A map $f : P \to \mathbb{Z}$ is a left enriched P-parition if for any $i <_P j$,

- $|f(i)| \leq |f(j)|$;
- $|f(i)| = |f(j)| \Rightarrow f(j) \geq 0$.

$P =$ 

Signed filters and singed antichains

$(P, <_P)$: a poset on $[d]$.

$$\mathcal{F}^{(e)}(P) := \left\{ (F, \varepsilon) \in \mathcal{F}(P) \times \{0, \pm 1\}^d : \varepsilon_i = \begin{cases} \pm 1 & (i \in \min(F)) \\ 1 & (i \in F \setminus \min(F)) \\ 0 & (i \notin F) \end{cases} \right\}$$

$$\mathcal{A}^{(e)}(P) := \left\{ (A, \varepsilon) \in \mathcal{A}(P) \times \{0, \pm 1\}^d : \varepsilon_i = \begin{cases} \pm 1 & (i \in A) \\ 0 & (i \notin A) \end{cases} \right\}$$

Remark

If P is naturally labeled, then

$$\mathcal{F}(P) \overset{\mathit{1{:}1}}{\longleftrightarrow} \{f : P\text{-partition with } f(i) \leq 1\}$$

$$\mathcal{F}^{(e)}(P) \overset{\mathit{1{:}1}}{\longleftrightarrow} \{f : \text{left enriched } P\text{-partition with } |f(i)| \leq 1\}$$

Two enriched poset polytopes

$(P, <_P)$: a poset on $[d]$.

- For $X \subset [d]$ and $\varepsilon \in \{0, \pm 1\}^d$, set $\mathbf{e}_X^\varepsilon := \sum_{i \in X} \varepsilon_i \mathbf{e}_i$.

Definition (Ohsugi-T)

The enriched order polytope of P is

$$\mathcal{O}_P^{(e)} := \mathrm{conv}\{\mathbf{e}_F^\varepsilon : (F, \varepsilon) \in \mathcal{F}^{(e)}(P)\}.$$

The enriched chain polytope of P is

$$\mathcal{C}_P^{(e)} := \mathrm{conv}\{\mathbf{e}_A^\varepsilon : (A, \varepsilon) \in \mathcal{A}^{(e)}(P)\}.$$

Remark

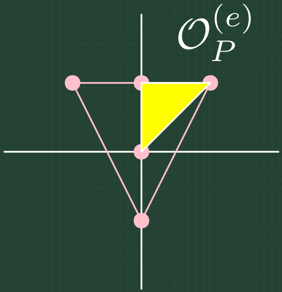
- $\mathbf{e}_F^\varepsilon$ *(resp.* $\mathbf{e}_A^\varepsilon$*) is not always a vertex of* $\mathcal{O}_P^{(e)}$ *(resp.* $\mathcal{C}_P^{(e)}$*).*
- $\mathcal{O}_P^{(e)}$ *and* $\mathcal{O}_{\overline{P}}^{(e)}$ *are not always isomorphic.*

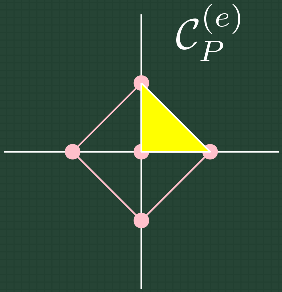
Example

$$P =$$

$$\mathcal{F}(P) = \{\emptyset, \{2\}, \{1,2\}\}$$

$$\mathcal{A}(P) = \{\emptyset, \{1\}, \{2\}\}$$





Left enriched order polynomial

$(P, <_P)$: a naturally labeled poset on $[d]$.
$\Omega_P^{(\ell)}(m) := |\{f$: left enriched P-partitions with $|f(i)| \leq m\}|$
: the left enriched order polynomial of P.

Theorem (Ohsugi-T)

$$L_{\mathcal{O}_P^{(e)}}(m) = L_{\mathcal{C}_P^{(e)}}(m) = \Omega_P^{(\ell)}(m).$$

Remark

For a (not necessarily naturally labeled) poset P on $[d]$,

$$L_{\mathcal{O}_P^{(e)}}(m) = L_{\mathcal{C}_P^{(e)}}(m) = L_{\mathcal{O}_{\overline{P}}^{(e)}}(m).$$

Corollary

$\Omega_P^{(\ell)}(m)$ *depends only on G_P.*

Palindromic polynomials and γ-positivity

$f(t) = \sum_{i=0}^{d} a_i t^i \in \mathbb{Z}_{>0}[t]$: a palindromic polynomial
i.e., $a_i = a_{d-i}$ for any $1 \leq i \leq \lfloor d/2 \rfloor$

Then there exists a unique expression

$$f(t) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i (1+t)^{d-2i}$$

$\gamma(t) := \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i \in \mathbb{Z}[t]$ is called the γ-polynomial of $f(t)$.

(RR) $f(t)$ is real-rooted if all roots of $f(t)$ are real.

(GP) $f(t)$ is γ-positive if $\gamma_i \geq 0$ for all i.

(UN) $f(t)$ is unimodal if $a_0 \leq \cdots \leq a_k \geq \cdots \geq a_d$ with some k.

In general, (RR) $\Rightarrow$ (GP) $\Rightarrow$ (UN). If $f(t)$ is γ-positive, then

$$f(t) \text{ is real-rooted } \iff \gamma(t) \text{ is real-rooted}$$

Gal Conjecture

A simplicial complex is called flag if for any minimall non-face $\mathcal{F}$, $|\mathcal{F}| \leq 2$.

Conjecture (Real Root Conjecture, disproved)

The h-polynomial of a flag triangulation of a sphere is real-rooted.

Gal found a counterexample for the Real Root Conjecture.

Conjecture (Gal Conjecture)

The h-polynomial of a flag triangulation of a sphere is γ-positive.

Conjecture (Nevo-Petersen Conjecture)

The γ-polynomial of the h-polynomial of a flag triangulation of a sphere coincides with the f-polynomial of a flag simplicial complex.

Ehrhart series and h^*-polynomials

$\mathcal{P} \subset \mathbb{R}^d$: a lattice polytopeof dimension d

$\text{Ehr}(\mathcal{P}, t) := 1 + \sum_{m=1}^{\infty} L_{\mathcal{P}}(m) t^m$: the Ehrhart series of $\mathcal{P}$.

$(1-t)^{d+1}\text{Ehr}(\mathcal{P}, t) = \sum_{i=0}^{d} h_i^* t^i =: h^*(\mathcal{P}, t)$: the h^*-polynomial of $\mathcal{P}$.

Remark

- *each $h_i^* \geq 0$ (Stanley).*
- $h_0^* = 1, h_1^* = |\mathcal{P} \cap \mathbb{Z}^d| - (d+1)$ *and* $h_d^* = |\text{int}(\mathcal{P}) \cap \mathbb{Z}^d|$.
- $h_0^* + \cdots + h_d^*$ *equals the normalized volume of* $\mathcal{P}$.

P-Eulerian polynomials

$(P, <_P)$: a naturally labeled poset on $[d]$.
A permutation $\pi = \pi_1 \cdots \pi_d$ is called a linear extension of P if $i <_P j \Rightarrow \pi_i < \pi_j$.
$\mathcal{L}(P)$: the set of linear extensions of P.
For $\pi \in \mathcal{L}(P)$, set

$$\mathrm{des}(\pi) := |\{1 \leq i \leq d-1 : \pi_i > \pi_{i+1}\}|.$$

$W_P(t) := \sum_{\pi \in \mathcal{L}(P)} t^{\mathrm{des}(\pi)}$: the P-Eulerian polynomial.

Theorem (Stanley)

$$h^*(\mathcal{O}_P, t) = h^*(\mathcal{C}_P, t) = W_P(t).$$

Palindromic P-Eulerian polynomials

Theorem (Hibi)

$h^*(\mathcal{O}_P, t)$ and $h^*(\mathcal{C}_P, t)$ are palindromic if and only if P is pure, i.e., every maximal chain of P has a same length.

Theorem (Reiner-Welker)

If P is pure, then $h^*(\mathcal{O}_P, t)$ and $h^*(\mathcal{C}_P, t)$ coincide with the h-polynomial of a flag triangulation of a sphere.

Theorem (Brändén)

If P is pure, then $h^*(\mathcal{O}_P, t)$ and $h^*(\mathcal{C}_P, t)$ are γ-positive.

Conjecture (Stanley)

$h^*(\mathcal{O}_P, t)$ and $h^*(\mathcal{C}_P, t)$ are unimodal.

Left peak polynomials

$(P, <_P)$: a naturally labeled poset on $[d]$.

Theorem (Ohsugi-T)

$h^*(\mathcal{O}_P^{(e)}, t)$ and $h^*(\mathcal{C}_P^{(e)}, t)$ always coincide with the h-polynomial of a flag triangulation of a sphere.

For $\pi \in \mathcal{L}(P)$ with $\pi_0 = 0$, set

$$\mathrm{peak}^{(\ell)}(\pi) := |\{1 \le i \le d-1 : \pi_{i-1} < \pi_i > \pi_{i+1}\}|.$$

$W_P^{(\ell)}(t) := \sum_{\pi \in \mathcal{L}(P)} t^{\mathrm{peak}^{(\ell)}(\pi)}$: the left peak polynomial of P.

Theorem (Ohsugi-T, Petersen, Stembridge)

The γ-polynomials of $h^*(\mathcal{O}_P^{(e)}, t)$ and $h^*(\mathcal{C}_P^{(e)}, t)$ equal $W_P^{(\ell)}(4t)$.

Theorem (Nevo-Petersen, Ohsugi-T)

$W_P^{(\ell)}(4t)$ coincides with the f-polynomial of a flag simplicial complex.