K-theory of toric hyperKähler manifolds

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Outline

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1. Motivation
2. Basic Construction
   - Construction of toric hyperKähler manifolds
   - Cohomology ring of toric hyperKähler manifolds
3. Our Results
   - K-ring of toric hyperKähler manifolds

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Earlier work

- Toric hyperKähler manifolds [BD00] were defined by Bielawski and Dancer who study their topology and geometry.
- The integral cohomology ring of toric hyperKähler manifolds was studied by Konno [K00] who gives a presentation for the cohomology ring.
- Recently Kuroki [KU11] studied equivariant cohomology ring of toric hyperKähler manifolds in relation to cohomological rigidity problem.
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Our aim

- To study K-theory of toric hyperKähler manifolds and toric hyperKähler varieties
- We would like to give a presentation of the K-ring using the combinatorics of the associated hyperplane arrangement
- Our earlier results on the K-ring of smooth projective toric varieties, quasitoric manifolds and torus manifolds used the combinatorics of fan or polytope.
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Notations

- $N := \mathbb{Z}^n$; $M \simeq Hom(N, \mathbb{Z})$
- $N' := \mathbb{Z}^m$; $M' := Hom(N', \mathbb{Z})$.
- Let $\{e_1, \ldots, e_m\}$ be a basis of N' and $\{e_1^*, \ldots, e_m^*\}$ be the dual basis of M'.
- $\hat{\alpha} := (\alpha_1, \ldots, \alpha_m) \in M'_{\mathbb{R}} := M' \otimes_{\mathbb{Z}} \mathbb{R}$.
- Let $v_1, \ldots, v_m$ be nonzero primitive vectors in N.

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Smooth hyperplane arrangements

- $H_i := \{x \in M_{\mathbb{R}} \mid \langle x, v_i \rangle + \alpha_i = 0\}$ is a codimension 1 affine subspace in $M_{\mathbb{R}}$ with a normal oriented vector v_i.
- $\mathcal{H} := \{H_1, \ldots, H_m\}$ is a hyperplane arrangement in $M_{\mathbb{R}}$.
- $\mathcal{H}$ is simple if each nonempty intersection of k hyperplanes has codimension k and if there are n hyperplanes with nonempty intersection
- $\mathcal{H}$ is smooth if $\mathcal{H}$ is simple and every n linearly independent vectors from $\{v_1, \ldots, v_m\}$ span N.

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Exact sequences

- Since $\mathcal{H}$ is smooth we have a surjective homomorphism $\rho : N' \longrightarrow N$ where $\rho(e_i) := v_i$ for $1 \leq i \leq m$.
- $N'' := \ker(\rho) \simeq \mathbb{Z}^{m-n}$ and $M'' = Hom(N'', \mathbb{Z})$.
- We get exact sequences of lattices:
- $$0 \longrightarrow N'' \stackrel{\iota}{\longrightarrow} N' \stackrel{\rho}{\longrightarrow} N \longrightarrow 0$$

  $$0 \longrightarrow M \stackrel{\rho^*}{\longrightarrow} M' \stackrel{\iota^*}{\longrightarrow} M'' \longrightarrow 0 \qquad (1)$$
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- We also get the corresponding exact sequences of vector spaces:
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$$0 \longrightarrow M_{\mathbb{R}} \xrightarrow{\rho^*_{\mathbb{R}}} M'_{\mathbb{R}} \xrightarrow{\iota^*_{\mathbb{R}}} M''_{\mathbb{R}} \longrightarrow 0$$
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HyperKalher structure on $\mathbb{H}^m$

- Consider $\mathbb{H}^m$ with 3 complex structures I, J, K induced by multiplication by i, j and k respectively satisfying the quaternionic relations.
- The diagonal torus $T' = (S^1)^m \subseteq Sp(m) \subseteq SO(4m)$ acts on $\mathbb{H}^m \simeq \mathbb{R}^{4m}$ preserving the Riemannian metric and the Kahler forms $\omega_I, \omega_J, \omega_K$ corresponding to the complex structures I, J and K respectively.
- $$\mu = (\mu_I, \mu_J, \mu_K) : \mathbb{H}^m \longrightarrow (M'_{\mathbb{R}})^3$$

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Definition of toric hyperKähler manifold

- This further induces an action of $G \hookrightarrow T'$ on $\mathbb{H}^m$ and $\mu_G := \iota_{\mathbb{R}}^* \circ \mu : \mathbb{H}^m \longrightarrow (M''_{\mathbb{R}})^3$ is the moment map for the G-action on $\mathbb{H}^m$.
- Since $\alpha \neq 0$, $(\alpha, 0, 0)$ is a regular value of μ_G.
- Since $\mathcal{H}$ is smooth, G acts freely on $\mu_G^{-1}(\alpha, 0, 0)$ and $\mu_G^{-1}(\alpha, 0, 0)/G$ is a smooth manifold of dimension $4n$.
- $X := \mu_G^{-1}(\alpha, 0, 0)/G$ is called toric hyperKähler manifold equipped with an action of the n-dimensional torus $T = T'/G$ which preserves the hyperKähler structure i.e the induced Riemannian metric and complex structures $I_\alpha, J_\alpha, K_\alpha$.

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Complex line bundles on X

- Let $\mathbb{C}_s$ be the 1-dimensional complex vector space with G-action induced by $G \hookrightarrow T' \xrightarrow{p_s} S^1$.
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Cohomology ring presentation

- Let X be a toric hyperKähler manifold $\mathcal{H} = \{H_1, \ldots, H_m\}$ be the associated *smooth hyperplane arrangement*
- Ideal J in $\mathbb{Z}[x_1, \ldots, x_m]$ generated by
  - $\prod_{s \in I} x_s$ whenever $\bigcap_{s \in I} H_s = \emptyset$, $I \subseteq [1, m]$
  - $\sum_{s=1}^{m} \langle u, v_s \rangle x_s$, $u \in M$.
- **Theorem**(Konno) There is an isomorphism of $\mathbb{Z}$-algebras $\phi : \mathbb{Z}[x_1, \ldots, x_m]/J \longrightarrow H^*(X; \mathbb{Z})$ that sends x_s to $c_1(L_s)$ for $1 \leq s \leq m$.

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Topological K-ring presentation

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- Although X is non-compact in general it is homotopy equivalent to its "core" Core(X) which is a finite union of compact toric submanifolds. (Core(X) is a strong deformation retract of X)

- We can apply the *Atiyah Hirzebruch spectral sequence* which degenerates in this setting since the integral odd cohomology vanishes.

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Cotangent bundle of complex projective space

Example

The cotangent bundle of the complex projective space $T^*(\mathbb{CP}^n)$ is a toric hyperKähler manifold associated to the hyperplane arrangement $\mathcal{H} = \{H_1, \ldots, H_n, H_{n+1}\}$ in $\mathbb{R}^n$ consisting of

$$H_j = \{(a_1, \ldots, a_n) \mid a_j = -1\}$$

for $1 \leq j \leq n$ and $H_{n+1} = \{(a_1, \ldots, a_n) \mid a_1 + \cdots + a_n = 1\}$.

K-ring of $T^*(\mathbb{CP}^n)$

Example

- J' is the ideal in $\mathbb{Z}[x_1, \ldots, x_{n+1}]$ generated by
  - the monomial $x_1 \cdot x_2 \cdots x_{n+1}$ since $I = [1, n+1]$ is the only subset such that $H_1 \cap \cdots \cap H_{n+1} = \emptyset$
  - and the n relations $(1 - x_j) - (1 - x_{n+1})$ for $1 \leq j \leq n$ corresponding to the basis $e_1^*, \ldots, e_n^*$.
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