On the double coinvariant rings of pseudo-reflection groups

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Flag manifolds and their cohomology rings

G: a cpt. 1-conn. Lie group of rank n, T: maximal torus of G,
$H^*(BT) \cong \mathbb{R}[x_1, \ldots, x_n]$, W: the Weyl group of G, $W \curvearrowright T$ and $H^*(BT)$.
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$T \curvearrowright G/T$, left multiplication

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Theorem 1 (Gorecky–Kottwicz–MacPherson '98)

$\mathrm{Im}\, i^* = \{f \in \mathrm{Map}(W, H^*(BT)) \mid f(w) - f(\sigma_\alpha w) \in (\alpha), \forall \alpha\text{: root}, \forall w \in W\}$
$\uparrow$ *reflection w.r.t.* α

$H_T^*(G/T;\mathbb{R}) \cong \mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[t_1, \ldots, t_n]/(f(x_1, \ldots, x_n) - f(t_1, \ldots, t_n))$
for all W-invariant polynomial f

The value of x_i at w is $x_i(w) = w \cdot t_i$.

Bruhat decomposition

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Mayer-Vietoris seq.

$$0 \longrightarrow H_T^i(S^2) \xrightarrow{i^*} H_T^i(\mathrm{pt}) \oplus H_T^i(\mathrm{pt}) \longrightarrow H_T^i(S^1) \longrightarrow 0,$$

where $H_T^*(S^1) = H^*(ET \times_T S^1) \cong H^*(BT)/(\alpha)$.

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$T_w X^{(2l(w))} = \bigoplus_{\alpha:\ \sigma_\alpha w < w} \mathbb{C}_\alpha$, the root space decomposition
When attaching C_w to $X^{(2l(w)-2)}$, if $\sigma_{\alpha_1} w, \ldots, \sigma_{\alpha_k} w \in X^{(2l(w)-2)}$, then the GKM condition $f(w) - f(\sigma_{\alpha_i} w) \in (\alpha_i)$ will appear.

pseudo-reflection groups

Definition 2

V: vector space/$\mathbb{C}$, $\sigma: V \to V$ is a *pseudo-reflection*
$\Leftrightarrow \operatorname{codim}_{\mathbb{C}} \ker(1_V - \sigma) = 1, \exists m \geq 2$ s.t. $\sigma^m = 1_V$.
A finite group W is a *pseudo-reflection group*
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Example 3

$\langle \zeta_m \rangle \cong \mathbb{Z}/m\mathbb{Z} \curvearrowright \mathbb{C}$, invariant elements $\cdots$ polynomial in x^m
coinvariant ring = $\mathbb{C}[x]/(x^m) \cong H^*(\mathbb{C}P^{m-1}; \mathbb{C})$

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Example 4

$W = G(4,1,2) \subset U(2)$ is the subgroup generated by
$\begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-1} \end{pmatrix}$. invariant elements $\cdots x_1^4 + x_2^4,\ x_1^4 x_2^4$

double coinvariant rings

Definition 5

The double coinvariant ring of a pseudo-reflection group W is $\mathbb{C}[V] \otimes_{\mathbb{C}[V]^W} \mathbb{C}[V]$.

$$i^* : \mathbb{C}[V] \otimes_{\mathbb{C}[V]^W} \mathbb{C}[V] \to \mathrm{Map}(W, \mathbb{C}[V]) \qquad a \otimes b \mapsto ((w \cdot a)b)_{w \in W}$$

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For each pseudo-reflection σ, choose $\alpha_\sigma \in V^*$.

λ_σ: the non-trivial eigenvalue of σ

Theorem 6 (McDaniel)

$$\mathrm{Im}\, i^* = \left\{ f \in \mathrm{Map}(W, \mathbb{C}[V]) \;\middle|\; \text{for } \forall \sigma, 1 \le i \le |\sigma|, \sum_{j=0}^{|\sigma|-1} \lambda_\sigma^{-ij} f(\sigma^j w) \in (\alpha_\sigma^i) \right\}$$

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This description does not admit “cell decompositions”.

Results

Let $H_\sigma = \ker \alpha_\sigma = \ker(1 - \sigma)$. The stabilizer of H_σ is a cyclic group. Choose a linear order on W, and restrict it on the orbits of stabilizers $Hw = \{w < \sigma^{i_2} w < \sigma^{i_3} w < \cdots < \sigma^{i_{|\sigma|}} w\}$ of $\exists$ generating reflections σ.

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$$\begin{vmatrix} f(w) & 1 \\ f(\sigma^{i_2} w) & 1 \end{vmatrix} \in (\alpha_\sigma), \begin{vmatrix} f(w) & 1 & 1 \\ f(\sigma^{i_2} w) & 1 & \lambda_\sigma^{i_2} \\ f(\sigma^{i_3} w) & 1 & \lambda_\sigma^{i_3} \end{vmatrix} \in (\alpha_\sigma^2), \ldots,$$

$$\begin{vmatrix} f(w) & 1 & 1 & \cdots & 1 \\ f(\sigma^{i_2} w) & 1 & \lambda_\sigma^{i_2} & \cdots & (\lambda_\sigma^{i_2})^{|\sigma|-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(\sigma^{i_{|\sigma|}} w) & 1 & \lambda_\sigma^{i_{|\sigma|}} & \cdots & (\lambda_\sigma^{i_{|\sigma|}})^{|\sigma|-2} \end{vmatrix} \in (\alpha_\sigma^{|\sigma|-1})$$