Basic Dolbeault cohomology of Canonical foliations on Moment-angle Manifolds

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Let Σ be a complete simplicial fan in $\mathbb{R}^n$ with generators $a_1, \ldots, a_m$ and $\mathcal{K} = \mathcal{K}_\Sigma$ be the underlying simplicial complex. Let $m - n$ be an even integer number. Define a moment-angle manifold corresponding to $\mathcal{K}$

$$\mathcal{Z}_\mathcal{K} := \bigcup_{\mathcal{I} \in \mathcal{K}} \Big(\prod_{i \in \mathcal{I}} \mathbb{D} \times \prod_{i \notin \mathcal{I}} \mathbb{S} \Big) \subseteq \mathbb{D}^m,$$

Any moment-angle manifold $\mathcal{Z}_\mathcal{K}$ is equipped with a natural action of the torus by coordinate-wise multiplication

$$T^m = \{(t_1, \ldots, t_m) \in \mathbb{C}^m \colon |t_i| = 1\}$$

The action is given by

$$T^m \times \mathcal{Z}_\mathcal{K} \to \mathcal{Z}_\mathcal{K} \colon ((t_1, \ldots, t_m), (z_1, \ldots, z_m)) \mapsto (t_1 z_1, \ldots, t_m z_m).$$

Define

$$U_\mathcal{K} = \bigcup_{\mathcal{I} \in \mathcal{K}} \Big(\prod_{i \in \mathcal{I}} \mathbb{C} \times \prod_{i \notin \mathcal{I}} \mathbb{C}^\times \Big) \subset \mathbb{C}^m.$$

Consider a linear map associated with Σ

$$q\colon \mathbb{R}^m \to \mathbb{R}^n, \quad \boldsymbol{e}_i \mapsto \boldsymbol{a}_i.$$

Let us choose a complex structure in $\operatorname{Ker} q$.This is equivalent to a choice of an $\frac{m-n}{2}$-dimensional complex subspace $\mathfrak{h} \subset \mathbb{C}^m$ satisfying the two conditions:

(a) the composite $\mathfrak{h} \hookrightarrow \mathbb{C}^m \xrightarrow{\mathrm{Re}} \mathbb{R}^m$ is injective;

(b) the composite $\mathfrak{h} \hookrightarrow \mathbb{C}^m \xrightarrow{\mathrm{Re}} \mathbb{R}^m \xrightarrow{q} \mathbb{R}^n$ is zero.

Consider the $\frac{m-n}{2}$-dimensional complex-analytic subgroup

$$H = \exp(\mathfrak{h}) \subset (\mathbb{C}^\times)^m.$$

Introduce a space

$$U_{\mathcal{K}} = \bigcup_{\mathcal{I}\in\mathcal{K}} \Big(\prod_{i\in\mathcal{I}} \mathbb{C} \times \prod_{i\notin\mathcal{I}} \mathbb{C}^\times\Big) \subset \mathbb{C}^m.$$

Theorem (Panov, Ustinovskiy; 2012)

The group H acts properly, freely and holomorphically on $U_{\mathcal{K}}$. Moreover, there is a T^m-equivariant diffeomorphism $U_{\mathcal{K}}/H \cong \mathcal{Z}_{\mathcal{K}}$.

Define a real Lie subalgebra and the corresponding Lie group

$$\mathfrak{r} = \mathrm{Ker}\, q = \mathrm{Re}(\mathfrak{h}) \subset \mathbb{R}^m = \mathfrak{t}, \qquad R = \exp(\mathfrak{r}) \subset T^m.$$

We also define a "complexification" of R as

$$R^{\mathbb{C}} = \exp(\mathfrak{r}^{\mathbb{C}}) = \exp(\mathrm{Ker}\, q^{\mathbb{C}}) \subset (\mathbb{C}^\times)^m.$$

The group $R^{\mathbb{C}}$ acts locally free and holomorphically on $U_{\mathcal{K}}$. This action defines a holomorphic foliation $\mathcal{F}_\Sigma$ on $U_{\mathcal{K}}$.
The holomorphic foliation $\mathcal{F}_\Sigma$ is mapped by the quotient projection $U_{\mathcal{K}} \to U_{\mathcal{K}}/H$ to a holomorphic foliation $\mathcal{F}_{\mathfrak{h}}$ of $\mathcal{Z}_{\mathcal{K}} \cong U_{\mathcal{K}}/H$ by the orbits of $R^{\mathbb{C}}/H \cong R$.

Fibration over a toric variety

When $\mathfrak{r} \subset \mathbb{R}^m$ is rational with respect to the lattice of the torus T^m the group R becomes a subtorus of T^m and the complete fan Σ becomes rational. Then we have

$$\mathcal{Z}_{\mathcal{K}}/R \cong V_\Sigma.$$

The canonical foliation on $\mathcal{Z}_{\mathcal{K}}$ by orbits of R turns into a locally trivial fibration over a toric variety V_Σ.

Basic de Rham cohomology

For a smooth manifold M with de Rham differential d and a foliation $\mathcal{F}$ on it we define a differential graded algebra of basic forms

$$(\Omega_{\mathcal{F}}(M), d) = \{\omega \in \Omega(M) \colon \iota_X \omega = L_X \omega = 0 \quad \text{for any } X \in T\mathcal{F}\}.$$

If the foliation $\mathcal{F}$ is induced by an action of a connected group G with Lie algebra $\mathfrak{g}$ then this algebra may be defined as

$$(\Omega_{\mathfrak{g}}(M), d) = \{\omega \in \Omega(M) \colon \iota_\xi \omega = L_\xi \omega = 0 \quad \text{for any } \xi \in \mathfrak{g}\}$$

where X_ξ is a vector field induced by an action of ξ and $\iota_\xi = \iota_{X_\xi}$, $L_\xi = L_{X_\xi}$.

Basic Dolbeault cohomology

For $\overline{\partial}$ differential on complex manifold M we have a bigraded differential algebra of basic forms with values in $\mathbb{C}$

$$(\Omega^{*,*}_{\mathcal{F}}(M; \mathbb{C}), \overline{\partial})$$

Theorem (Ishida, K., Panov, 2018)

There is an isomorphism of algebras:

$$H^*_{\mathcal{F}_\mathfrak{h}}(\mathcal{Z}_\mathcal{K}) \cong \mathbb{R}[v_1, \ldots, v_m]/(\mathcal{I}_\mathcal{K} + \mathcal{J}),$$

where $\mathcal{I}_\mathcal{K}$ is the Stanley–Reisner ideal of simplicial complex $\mathcal{K}$, generated by monomials

$$v_{i_1} \ldots v_{i_k}, \quad \textit{with} \quad \{i_1, \ldots, i_k\} \notin \mathcal{K},$$

and $\mathcal{J}$ is the ideal generated by the linear forms

$$\sum_{i=1}^{m} \langle \boldsymbol{u}, \boldsymbol{a}_i \rangle v_i, \quad \boldsymbol{u} \in \mathfrak{g}' = (\mathfrak{t}/\mathfrak{r})^*.$$

Definition

A holomorphic foliation $(M, \mathcal{F})$ is called transverse Kähler if it is *homologically orientable* and there exists a transverse 2-form $\omega_{\mathcal{F}}$ such that

- $d\omega_{\mathcal{F}} = 0$;
- $\omega_{\mathcal{F}}(JX, JY) = \omega(X, Y)$;
- $\omega_{\mathcal{F}}(JV, JV) \geqslant 0$.

Theorem (Ishida, 2015)

The canonical foliation $\mathcal{F}_{\mathfrak{h}}$ on $\mathcal{Z}_{\mathcal{K}}$ is transverse Kähler if and only if the fan Σ is polytopal.

Theorem (Ishida, 2018)

Let $\mathcal{F}_{\mathfrak{h}}$ be a transverse Kähler foliation. Than there is a Hodge decomposition

$$H^r_{\mathcal{F}_{\mathfrak{h}}}(\mathcal{Z}_{\mathcal{K}}; \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}_{\mathcal{F}_{\mathfrak{h}}}(\mathcal{Z}_{\mathcal{K}}).$$

Moreover, there is an isomorphism of algebras:

$$H^{*,*}_{\mathcal{F}_{\mathfrak{h}}}(\mathcal{Z}_{\mathcal{K}}) \cong \mathbb{R}[v_1, \dots, v_m]/(I_{\mathcal{K}} + J), \quad v_i \in H^{1,1}_{\mathcal{F}_{\mathfrak{h}}}(\mathcal{Z}_{\mathcal{K}}).$$

Theorem (Lin, Yang, 2019)

Let G be a compact torus, and let $(M, \mathcal{F})$ be a transverse Kähler foliation. Suppose that there is a holomorphic Hamiltonian action of G on $(M, \mathcal{F})$. Let X_G be a fixed-leaf set of this action. Then $H^{p,q}(M, \mathcal{F}) = 0$ for $|p - q| > codim(\mathcal{F}|_{X_G})$.

Remark

This generalizes Ishida's result since in case of a moment-angle manifold $codim(\mathcal{F}_{\mathfrak{h}}|_{X_{T^m}}) = 0$ because a fixed-leaf set consists of finite number of leaves.

Theorem (K., Panov, 2019)

There is a Hodge decomposition

$$H^r_{\mathcal{F}_\mathfrak{h}}(\mathcal{Z}_\mathcal{K};\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}_{\mathcal{F}_\mathfrak{h}}(\mathcal{Z}_\mathcal{K}).$$

Moreover, there is an isomorphism of algebras:

$$H^{*,*}_{\mathcal{F}_\mathfrak{h}}(\mathcal{Z}_\mathcal{K}) \cong \mathbb{R}[v_1,\ldots,v_m]/(I_\mathcal{K}+J), \quad v_i \in H^{1,1}_{\mathcal{F}_\mathfrak{h}}(\mathcal{Z}_\mathcal{K}).$$

Corollary

The analogous result also holds for complex manifolds with maximal torus action, which also include LVMB-manifolds.

Remark

To show this general result we introduce a concept of *Fujiki foliations* and prove that canonical foliation is always of this type. Although this is weaker than being transverse Kähler it also provides the Hodge decomposition.

Definition

A holomorphic foliation $(M, \mathcal{F})$ on a compact manifold M is a *Fujiki foliation* if it is homologically orientable and there exists a holomorphic foliated surjective map

$$f\colon (M', \mathcal{F}') \to (M, \mathcal{F}),$$

where $(M', \mathcal{F}')$ is a transverse Kähler foliation.

Lemma (3)

The map f as above induces an injection at the level of basic Dolbeault cohomology. As a consequence, if $(M, \mathcal{F})$ is a Fujiki foliation then its basic Dolbeault cohomology ring admits a Hodge decomposition

$$H^r_{\mathcal{F}}(M; \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}_{\mathcal{F}}(M).$$

We show that any canonical foliation $(\mathcal{Z}_{\mathcal{K}}, \mathcal{F}_{\mathfrak{h}})$ is Fujiki. What is more, we construct a holomorphic foliated surjection

$$f\colon (\mathcal{Z}_{\mathcal{K}'}, \mathcal{F}_{\mathfrak{h}'}) \to (\mathcal{Z}_{\mathcal{K}}, \mathcal{F}_{\mathfrak{h}}),$$

where $\mathcal{K}'$ corresponds to a polytope.

Let $\tau \in \Sigma$ be a $\boldsymbol{k}$-dimensional cone ($\boldsymbol{k} > 1$) generated by $a_1, \ldots, a_{\boldsymbol{k}}$, and denote the corresponding simplex by $I = \{1, \ldots, \boldsymbol{k}\} \in \mathcal{K}$.
Let $\mathcal{K}_\tau$ be a stellar subdivision $\mathrm{st}(\mathcal{K}, I)$ of $\mathcal{K}$ at I. We put Σ_τ to be a fan with generators $a_1, \ldots, a_m, a_0 = a_1 + \ldots + a_k$ such that the underlying simplicial complex is equal to $\mathcal{K}_\tau$.

Consider a projection corresponding to the fan Σ_τ:

$$q_\tau : \mathbb{R}^{m+1} \to \mathbb{R}^n, \quad e_0 \mapsto a_1 + \cdots + a_k, \; e_i \mapsto a_i, \; \boldsymbol{i} = 1, \ldots, \boldsymbol{k}.$$

We have

$$\mathrm{Ker}\, q_\tau = \langle e_1 + \cdots + e_k - e_0, \mathrm{Ker}\, q \rangle,$$

Then $\mathcal{F}_{\Sigma_\tau}$ is a foliation on $U_{\mathcal{K}_\tau}$ by the orbits of $R_\tau^{\mathbb{C}} = \exp(\mathrm{Ker}\, q_\tau^{\mathbb{C}})$.

Construction

Consider a holomorphic surjective map

$$f_\tau \colon \mathbb{C}^{m+1} \to \mathbb{C}^m, \quad (z_0, z_1, \ldots, z_m) \mapsto (z_0 z_1, \ldots, z_0 z_k, z_{k+1}, \ldots, z_m).$$

It restricts to a holomorphic foliated surjective map

$$f_\tau \colon (U_{\mathcal{K}_\tau}, \mathcal{F}_{\Sigma_\tau}) \to (U_{\mathcal{K}}, \mathcal{F}_\Sigma).$$

Remark

When Σ is a rational fan, both $R^{\mathbb{C}} \subset (\mathbb{C}^\times)^m$ and $R^{\mathbb{C}}_\tau \subset (\mathbb{C}^\times)^{m+1}$ are closed subgroups and the map $f_\tau \colon U_{\mathcal{K}_\tau} \to U_{\mathcal{K}}$ covers the standard blow-down map $V_{\Sigma_\tau} \to V_\Sigma$ of the quotient toric varieties $V_{\Sigma_\tau} = U(\mathcal{K}_\tau)/R^{\mathbb{C}}_\tau$ and $V_\Sigma = U_{\mathcal{K}}/R^{\mathbb{C}}$.

Proof of the theorem

Lemma (Adiprasito and Izmestiev, 2014)

If $\mathcal{K}$ is PL sphere then there exists $\boldsymbol{k'}$ such that $\mathrm{bs}^{k'}\mathcal{K}$ ($\boldsymbol{k'}$-th barycentric subdivision) is polytopal.

The barycentric subdivision is obtained as the composition of stellar subdivisions at all simplices of $\mathcal{K}$

$$\mathrm{bs}^{\mathbf{1}}\mathcal{K} = \mathrm{st}(I_1, \mathrm{st}(I_2, \ldots, \mathrm{st}(I_{\boldsymbol{N}}, \mathcal{K})\ldots)).$$

Therefore, we obtain a holomorphic foliated surjection

$$f_{\mathrm{bs}} \colon (U_{\mathrm{bs}^{\mathbf{1}}\mathcal{K}}, \mathcal{F}_{\mathrm{bs}^{\mathbf{1}}\Sigma}) \to (U_{\mathcal{K}}, \mathcal{F}_{\Sigma})$$

as the composition $f_{\tau_{\mathbf{1}}} \circ \cdots \circ f_{\tau_N}$, where $\tau_{\boldsymbol{j}}$ is the cone corresponding to the simplex $I_{\boldsymbol{j}}$.

Applying lemma and the composition of maps described above we obtain a foliated surjective map

$$f \colon (U_{\mathcal{K}'}, \mathcal{F}_{\Sigma'}) \to (U_{\mathcal{K}}, \mathcal{F}_{\Sigma})$$

such that Σ' is a polytopal fan.

Let $q^{\mathbb{C}}\colon \mathbb{C}^m \to \mathbb{C}^n$ and $q'^{\mathbb{C}}\colon \mathbb{C}^{m'} \to \mathbb{C}^n$ be the linear maps corresponding to Σ and Σ' respectively.
We have

$$\operatorname{Ker} q'^{\mathbb{C}} = \langle V, \operatorname{Ker} q^{\mathbb{C}} \rangle,$$

where V is the subspace generated by the vectors $e_{j_0} - e_{j_1} - \cdots - e_{j_s}$ corresponding to all stellar subdivisions in the sequence.

We pick an arbitrary half-dimensional complex subspace $\mathfrak{h}_0 \subset V$ such that its projection onto the real part $\mathbb{R}^{m'}$ is injective. Then the vector space $\mathfrak{h}_0 \oplus \mathfrak{h}$ provides a complex structure on $\mathcal{Z}_{\mathcal{K}'}$.
It is now clear that we have the following commutative diagram

$$\begin{array}{ccc} (U_{\mathcal{K}'}, \mathcal{F}_{\Sigma'}) & \xrightarrow{f} & (U_{\mathcal{K}}, \mathcal{F}_{\Sigma}) \\ \downarrow \pi_{\Sigma'} & & \downarrow \pi_{\Sigma} \\ (\mathcal{Z}_{\mathcal{K}'}, \mathcal{F}_{\mathfrak{h}_0 \oplus \mathfrak{h}}) & \xrightarrow{f_{\mathcal{Z}}} & (\mathcal{Z}_{\mathcal{K}}, \mathcal{F}_{\mathfrak{h}}) \end{array}$$

The map $f_{\mathcal{Z}}$ is a holomorphic and surjective. Now applying lemma 3, Ishida's theorem and the description of the basic de Rham cohomology of $\mathcal{Z}_{\mathcal{K}}$ we get the result.

Thank you for your attention!