Betti numbers for Hamiltonian circle actions with isolated fixed points

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Toric Topology in Okayama
November 18, 2019

I. Hamiltonian circle actions

Definition. A **symplectic form** ω on a manifold M is a differential 2-form such that

- $d\omega = 0$,
- $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is non-degenerate for every $p \in M$.

$\rightsquigarrow \quad \dim_{\mathbb{R}} M$ is even.

Equivalently, ω is symplectic if and only if

- $[\omega] \in H^2(M;\mathbb{R})$,
- $\omega^n := \underbrace{\omega \wedge \cdots \wedge \omega}_{n}$ is nowhere vanishing on M

Symplectic form is a "**device**" producing

$$\text{function } H : M \to \mathbb{R} \quad \rightsquigarrow \quad \text{vector field } X_H \text{ on } M$$

such that each integral curves of X_H obey the law of the "**conservation of energy H**"

By non-degeneracy of ω, we have

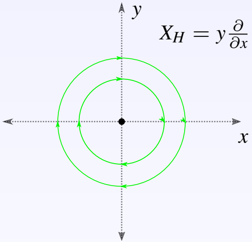
$$\begin{array}{cccccc} \omega & : & TM & \xrightarrow{\simeq} & T^*M & \\ & & X & \mapsto & \omega(X, \cdot) & \\ & & & & & \\ & & X_H & \leftarrow\!\shortmid & dH & \text{(exact 1-form)} \end{array}$$

Definition: For any $H : M \to \mathbb{R}$, we call X_H a **Hamiltonian vector field** where

$$dH = \omega(X_H, \cdot).$$

(Law of the conservation of H : $dH(X_H) = \omega(X_H, X_H) = 0$.)

Example :



$$X_H = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

$$\xrightarrow{H}$$

$$(x, y) \mapsto \frac{1}{2}(x^2 + y^2)$$

$\mathbb{R}$

$$dx \wedge dy\,(X_H, \cdot) = xdx + ydy = dH$$

In this case, we say that H generates a **Hamiltonian S^1-action**.

Definition : Assume S^1 acts on a **compact** symplectic manifold (M, ω). Let

$$\underline{X}_p := \frac{d}{dt}\Big|_{t=0} e^{it} \cdot p$$

The action is called **Hamiltonian** if $\underline{X}$ is Hamiltonian, i.e.,

$$dH = \omega(\underline{X}, \cdot)$$

for some $H : M \to \mathbb{R}$ (called a **moment map**).

Useful facts :

- Since $dH = \omega(\underline{X}, \cdot)$, we have that

$$dH(p) = 0 \quad \Leftrightarrow \quad \underline{X}_p = 0 \text{ (i.e., } p \text{ is a fixed point).}$$

  Thus, p is a **critical point** of H if and only if p is a **fixed point**.

- For each fixed point p, the action locally looks like

$$t \cdot (z_1, \cdots, z_n) = (t^{k_1} z_1, \cdots, t^{k_n} z_n), \qquad H(\mathbf{z}) = \frac{1}{2} \sum_{i=1}^{n} k_i |z_i|^2 + H(p)$$

  Thus, H is a **Morse (or Morse-Bott) function** and

$$\mathrm{ind}(p) = 2 \times (\# \text{ negative } k_i\text{'s})$$

  This implies that

$$\dim H^{2k}(M) = \dim H_{2k}(M) = \# \text{ fixed points of index } 2k$$

Example : Consider $M = \mathbb{P}^3$ with

$$t \cdot [z_0, \cdots, z_3] = [z_0, tz_1, t^2 z_2, t^3 z_3]$$

Then the weights at each fixed points are

- $[1,0,0,0]$: wt = $(1,2,3)$ $\qquad H^0(M) = \mathbb{Z}$
- $[0,1,0,0]$: wt = $(-1,1,2)$ $\qquad H^2(M) = \mathbb{Z}$
- $[0,0,1,0]$: wt = $(-2,-1,1)$ $\qquad H^4(M) = \mathbb{Z}$
- $[0,0,0,1]$: wt = $(-3,-2,-1)$ $\qquad H^6(M) = \mathbb{Z}$

Suppose (M^{2n}, ω) be a closed Hamiltonian S^1-manifold with only **isolated** fixed points.

Conjecture 1. (M, ω) is **Kähler.**

Conjecture 2. **(Hard Lefschetz property)** The map

$$[\omega]^{n-k} : H^k(M;\mathbb{R}) \to H^{2n-k}(M;\mathbb{R})$$

is an isomorphism for every $0 \leq k \leq n$.

Conjecture 3. The sequence $b_0(M) = 1, b_2(M), \cdots, b_{2n}(M)$ is **unimodal**, i.e.,

$$b_{k-2} \leq b_k \quad \text{for } {}^{\forall} k \leq n.$$

Conj. 1 $\Rightarrow$ Conj. 2 $\Rightarrow$ Conj. 3

Note : Conj. 2 and 3 are about **topology** of (M, ω). (algebraic structure of $H^*(M)$)

(Delzant 1988) There is one-to-one correspondence

$$\{\text{compact Ham. } T^n\text{-mfds } (M^{2n}, \omega)\} \quad \overset{1:1}{\longleftrightarrow} \quad \{\text{integral simple polytopes}\}$$
$$(M, \omega) \qquad\qquad\qquad\qquad H(M)$$

Moreover, every such manifold (M, ω) is **Kähler** (toric variety)

(Karshon 1999) : 4-dim. compact Hamiltonian S^1-manifolds are **Kähler**

(Tolman 2007 & McDuff 2008) : There are exactly four 6-dim. monotone Hamiltonian S^1-manifolds with $b_2(M) = 1$ and they are **Kähler**

(Knop 2010) Classification of multiplicity-free Hamiltonian G-spaces (**Proof of Delzant's conjecture**)

(Karshon - Tolman 2014) Classification of complexity-one spaces (under assumption : every fixed component is two-dimensional)

(**C. 2019**) Classification of 6-dim. monotone semifree S^1-actions (There are finitely many and all such manifolds are **Kähler**)

II. Equivariant Cohomology

Definition : Consider $S^{2n-1} \subset \mathbb{C}^n$ with a **free** S^1-action :

$$t \cdot (z_1, \cdots, z_n) = (tz_1, \cdots, tz_n).$$

Let $S^\infty := \lim_{n\to\infty} S^{2n-1}$ be the direct limit with the induced topology and S^1-action.
(It is known that S^∞ is contractible.)

$$H^*_{S^1}(M) := H^*(M \times_{S^1} S^\infty)$$

is called an **S^1-equivariant cohomology**

Fiber bundle structure :

$$\pi : M \times S^\infty \to S^\infty \quad \Rightarrow \quad \pi : M \times_{S^1} S^\infty \to \mathbb{P}^\infty = S^\infty / S^1$$

Thus $M \times_{S^1} S^\infty$ is an M-bundle over $\mathbb{P}^\infty$.

Theorem (equivariant formality) : $H^*_{S^1}(M) \cong H^*(M) \otimes H^*(\mathbb{P}^\infty)$
(i.e., free $H^*(\mathbb{P}^\infty)$-module)

Note :

- $H^*(\mathbb{P}^\infty) \cong \mathbb{R}[x]. \quad (\deg x = 2)$
- $\dim H^{2k}_{S^1}(M) = b_0 + b_2 + \cdots + b_{2k} \qquad (b_{2k} = \dim H^{2k}(M))$

Ring structure of $H^*_{S^1}(M)$:

For each fixed point p and an inclusion $i_p : p \hookrightarrow M$,

$$i_p : p \times_{S^1} S^\infty \hookrightarrow M \times_{S^1} S^\infty$$

induces a map (called **restriction to p**)

$$i_p^* : H^*_{S^1}(M) \to H^*_{S^1}(p) = H^*(\mathbb{P}^\infty) \cong \mathbb{R}[x]$$

Collecting all such maps, we have

$$\begin{array}{rcccccc} i^* & : & H^*_{S^1}(M) & \to & \bigoplus_{p \in M^{S^1}} H^*_{S^1}(p) & = & \bigoplus_{p \in M^{S^1}} \mathbb{R}[x] \\ & & \alpha & \mapsto & i^*(\alpha) & := & (i_p^*(\alpha))_{p \in M^{S^1}}. \end{array}$$

Theorem (Kirwan) The map i^* is a ring monomorphism.

Canonical basis of $H^*_{S^1}(M)$:

Recall that $H^*_{S^1}(M) \cong H^*(M) \otimes H^*(\mathbb{P}^\infty)$ is a **free** $H^*(\mathbb{P}^\infty)$-module.

Theorem (McDuff-Tolman) There exists a basis

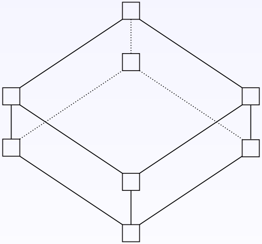
$$\{\alpha_p\}_{p \in M^{S^1}} \qquad \deg \alpha_p = \operatorname{ind}(p)$$

of $H^*_{S^1}(M)$ such that

- $i_q^*(\alpha_p) = 0$ when $p \neq q$ and either $\begin{cases} H(q) \leq H(p) \text{ or} \\ \operatorname{ind}(q) \leq \operatorname{ind}(p) \end{cases}$
- $i_p^*(\alpha_p) = x^{\lambda(p)}$ where $\lambda(p) := \frac{1}{2}\operatorname{ind}(p)$.

Visualizing canonical basis : $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\omega = a\omega_{\mathrm{FS}} \oplus b\omega_{\mathrm{FS}} \oplus c\omega_{\mathrm{FS}}$

$$t \cdot ([z_0, w_0], [z_1, w_1], [z_2, w_2]) = ([z_0, tw_0], [z_1, tw_1], [z_2, tw_2])$$



$M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

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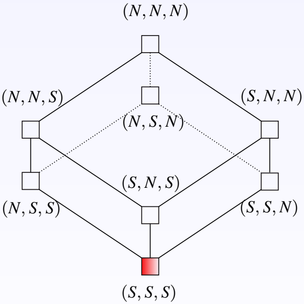
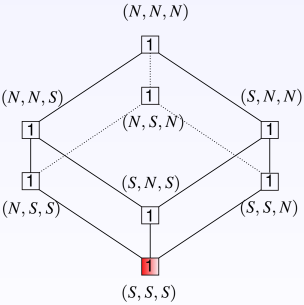


E.g. Weights at (S, S, S) : $(1, 1, 1) \quad \Rightarrow \quad \text{index} = 0$

Visualizing canonical basis : $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\omega = a\omega_{\mathrm{FS}} \oplus b\omega_{\mathrm{FS}} \oplus c\omega_{\mathrm{FS}}$

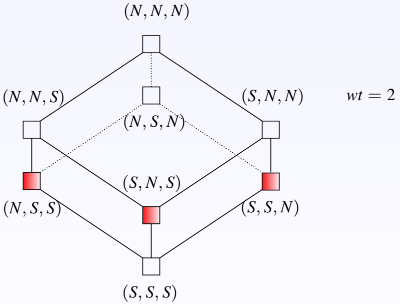
$$t \cdot ([z_0, w_0], [z_1, w_1], [z_2, w_2]) = ([z_0, tw_0], [z_1, tw_1], [z_2, tw_2])$$



Canonical class $\alpha_{S,S,S} \in H^0_{S^1}(M)$

Visualizing canonical basis : $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\omega = a\omega_{\mathrm{FS}} \oplus b\omega_{\mathrm{FS}} \oplus c\omega_{\mathrm{FS}}$

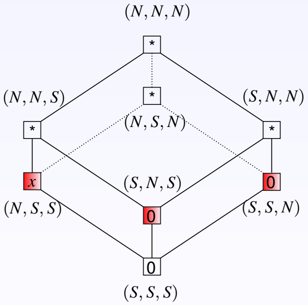
$$t \cdot ([z_0, w_0], [z_1, w_1], [z_2, w_2]) = ([z_0, tw_0], [z_1, tw_1], [z_2, tw_2])$$



E.g. Weights at (N, S, S) : $(-1, 1, 1) \quad \Rightarrow \quad \text{index} = 2$

Visualizing canonical basis : $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\omega = a\omega_{\text{FS}} \oplus b\omega_{\text{FS}} \oplus c\omega_{\text{FS}}$

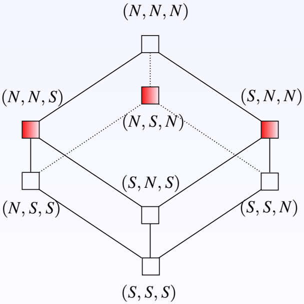
$$t \cdot ([z_0, w_0], [z_1, w_1], [z_2, w_2]) = ([z_0, tw_0], [z_1, tw_1], [z_2, tw_2])$$



Canonical class $\alpha_{N,S,S} \in H^2_{S^1}(M)$

Visualizing canonical basis : $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\omega = a\omega_{\mathrm{FS}} \oplus b\omega_{\mathrm{FS}} \oplus c\omega_{\mathrm{FS}}$

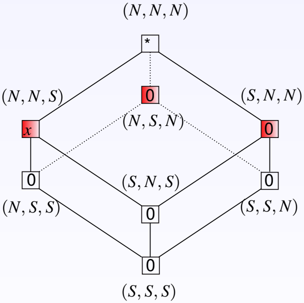
$$t \cdot ([z_0, w_0], [z_1, w_1], [z_2, w_2]) = ([z_0, tw_0], [z_1, tw_1], [z_2, tw_2])$$



E.g. Weights at (N, N, S) : $(-1, -1, 1) \quad \Rightarrow \quad \text{index} = 4$

Visualizing canonical basis : $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\omega = a\omega_{\mathrm{FS}} \oplus b\omega_{\mathrm{FS}} \oplus c\omega_{\mathrm{FS}}$

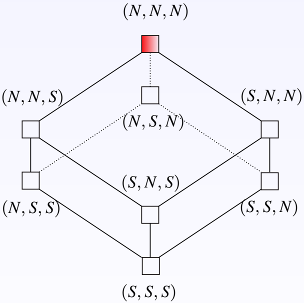
$$t \cdot ([z_0, w_0], [z_1, w_1], [z_2, w_2]) = ([z_0, tw_0], [z_1, tw_1], [z_2, tw_2])$$



Canonical class $\alpha_{N,N,S} \in H^4_{S^1}(M)$

Visualizing canonical basis : $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\omega = a\omega_{FS} \oplus b\omega_{FS} \oplus c\omega_{FS}$

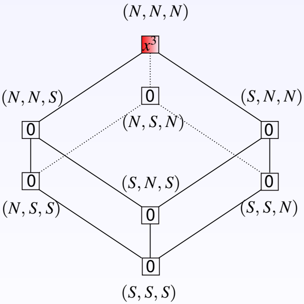
$$t \cdot ([z_0, w_0], [z_1, w_1], [z_2, w_2]) = ([z_0, tw_0], [z_1, tw_1], [z_2, tw_2])$$



E.g. Weights at (N, N, N) : $(-1, -1, -1) \quad \Rightarrow \quad \text{index} = 6$

Visualizing canonical basis : $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\omega = a\omega_{\mathrm{FS}} \oplus b\omega_{\mathrm{FS}} \oplus c\omega_{\mathrm{FS}}$

$$t \cdot ([z_0, w_0], [z_1, w_1], [z_2, w_2]) = ([z_0, tw_0], [z_1, tw_1], [z_2, tw_2])$$



Canonical class $\alpha_{N,N,N} \in H^6_{S^1}(M)$

Localization Theorem (Atiyah-Bott, Berline-Vergne): Let M be a compact S^1-manifold with only isolated fixed points. For any $\alpha \in H^*_{S^1}(M)$, we have

$$\int_M \alpha = \sum_{p \in M^{S^1}} \frac{i_p^*(\alpha)}{\prod_{i=1}^n \ w_i(p)x}$$

where $w_1(p), \cdots, w_n(p)$ is the weights of the S^1-action at p. **In particular,**

$$\sum_{p \in M^{S^1}} \frac{i_p^*(\alpha)}{\prod_{i=1}^n \ w_i(p)} = 0$$

for every α with $\deg \alpha < 2n$.

Example : For $\alpha_{N,N,S} \in H^4_{S^1}(M)$, we have

$$0 = \frac{x^2}{1} + \frac{*}{-1}$$

So, $* = x^2$.

Equivariant symplectic class: For a fixed moment map H, there exists

$$[\omega_H] \in H^2_{S^1}(M) \qquad i_p^*([\omega_H]) = H(p)x \in \mathbb{R}[x]$$

If we choose H such that $H_{\max} = 0$, then

$$[\omega_H]|_p = H(p)x < 0$$

for every $p \neq \max$.

III. Main Theorem

Theorem (C.) Let (M, ω) be a compact Hamiltonian S^1-manifold with isolated fixed points.

1. For $\dim M = 8n$ or $8n + 4$,

$$b_2 + \cdots + b_{2+4(n-1)} \leq b_4 + \cdots + b_{4+4(n-1)}$$

   E.g. If $\dim M = 8$, then $b_2 \leq b_4$ (so that unimodality holds)

2. If the moment map is index-increasing, then the unimodality conjecture holds.

Exercise : Show that there is no compact Hamiltonian S^1-manifolds with isolated fixed points such that

$$b_0 = b_8 = 1, \quad b_2 = b_6 = 2, \quad b_4 = 1$$

(That is, the sequence of Betti numbers is (1,2,1,2,1) which is not unimodal.)

Proof: Let

p_1, p_2 : fixed points of index 2, q of index 4, r_1, r_2 of index 6, and s of index 8

- Consider $\alpha_{p_1}, \alpha_{p_2} \in H^2_{S^1}(M)$. Then there exists some nonzero

$$\alpha := a\alpha_{p_1} + b\alpha_{p_2} \in H^2_{S^1}(M) \quad \text{s.t.} \quad i_q^*(\alpha) = 0$$

  Then α only survives at p_1, p_2, r_1, r_2,and s
- **(Symplectic class)** There exists $[\omega] \in H^2_{S^1}(M)$ such that $i_p^*([\omega]) = H(p)x$.
- Adjusting H so that $H(s) = 0$, we obtain a contradiction

$$0 = \int_M \alpha^2 \cdot [\omega] = \sum_p \frac{i_p^*(\alpha)^2 \cdot H(p)x}{w_i(p)x} = \sum \frac{-}{-} \neq 0$$

Proof of Theorem : Assume that $\dim M = 8n$. Note that

- $\dim_{\mathbb{R}} H^{4n-2}_{S^1}(M;\mathbb{R}) \cong b_0 + b_2 + \cdots b_{4n-2}$
- $\{u^{2n-1-\lambda(p)} \cdot \alpha_p \mid p \in M^{S^1},\ \lambda(p) \leq 2n-1\}$ is a basis of $H^{4n-2}_{S^1}(M;\mathbb{R})$ (as an $\mathbb{R}$-vector space)

Assume $\dim = 8n$ and

$$b_2 + \cdots + b_{2+4(n-1)} > b_4 + \cdots + b_{4+4(n-1)}$$

Now, consider the following map

$$\begin{array}{cccc} \Phi & : & H^{4n-2}_{S^1}(M;\mathbb{R}) & \rightarrow & \left(\mathbb{R}^{b_0} \oplus \mathbb{R}^{b_4} \oplus \cdots \oplus \mathbb{R}^{b_{4(n-1)}}\right) \oplus \left(\mathbb{R}^{b_{4n}} \oplus \cdots \oplus \mathbb{R}^{b_{8n-4}}\right) \\ & & \alpha & \mapsto & (\alpha_0, \cdots, \alpha_{4n-4}, \alpha_{4n}, \cdots, \alpha_{8n-4}) \end{array}$$

with the identification

$$\mathbb{R}^{b_{4i}} = \bigoplus_{\mathrm{ind}(p)=4i} \mathbb{R} \cdot u^{2n-1} \quad \text{and} \quad \alpha_{4i} := (\alpha|_p)_{\mathrm{ind}(p)=4i} \in \bigoplus_{\mathrm{ind}(p)=4i} \mathbb{R} \cdot u^{2n-1}$$

By dimensional reason, Φ has a **non-trivial kernel** $\alpha \in H^2_{S^1}(M)$. Then $\alpha^2 \cdot [\omega_H] \in H^{8n-4}(M)$ only survives at fixed points of index $2, 6, \cdots, 8n-6, 8n-2$. This contradicts the localization theorem.