# Combinatorics of torus actions of complexity one

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# General assumptions

- $X = X^{2n}$ : smooth closed connected $2n$-manifold;
- $T^k \circlearrowright X$ : effective action of the compact torus;
- $0 < \#X^T < \infty$ : fixed points exist and they are isolated;
- $n - k \geqslant 0$ is called the complexity of the action.

**Definition**

Action is called equivariantly formal if $H^{\mathrm{odd}}(X) = 0$ (equivalently, $H^*(X) \cong H^*_T(X) \otimes_{H^*(BT)} \mathbb{Z}$).

# Motivation: actions of complexity 0

Complexity 0: $T^n \circlearrowright X^{2n}$.

- $P^n = X/T$ is a manifold with corners.
- (Masuda–Panov'06) $X$ is equivariantly formal $\Leftrightarrow$ all faces of $P$ are acyclic: $\forall F \subseteq P : \widetilde{H}_*(F) = 0$. For eq.formal actions we have:
- Even Betti numbers are expressed from combinatorics of $P$:

$$\sum_{i=0}^{n} \beta_{2i}(X) t^{2i} = \sum_{i=0}^{n} f_i t^{2n-2i} (1 - t^2)^i,$$

  where $f_i$ is the number of $i$-dimensional faces of $P$.
- $H_T^*(X) \cong \mathbb{Z}[P]$ and $H^*(X) \cong \mathbb{Z}[P]/(l.s.o.p.)$, where $\mathbb{Z}[P]$ is the face ring of $P$.

# Plan of today's talk

Complexity 1: $T^{n-1} \circlearrowright X^{2n}$.

- We consider only actions in general position.
- $Q^{n+1} = X/T$ is a closed topological manifold. But there is combinatorial structure: the sponge $Z$ and its faces.
- (A.–Masuda'19) Criterion of equivariant formality in terms of the orbit space, faces, and sponge.
- Even Betti numbers are computed from the combinatorics of the sponge $Z$. Examples.
- Face algebra? This is a problem.

# Actions of complexity one in general position

**Definition**

Let $\alpha_{x,1}, \ldots, \alpha_{x,n} \in \mathrm{Hom}(T^{n-1}, T^1) \cong \mathbb{Z}^{n-1}$ be the weights of the tangent representation $\tau_x X$ for a fixed point $x \in X^T$. If, for any $x \in X^T$, any $n-1$ of $\{\alpha_{x,i}\}$ are linearly independent over $\mathbb{Q}$, we call the action $T^{n-1}$ on $X$ an action in general position.

Examples: $T^3 \circlearrowright G_{4,2}$, $T^2 \circlearrowright F_3$, $T^3 \circlearrowright \mathbb{H}P^2$, $T^2 \circlearrowright S^6 = G_2/SU(3)$.

From now on, it will be assumed that all actions are of complexity one and in general position.

# Orbit spaces and sponges

The orbit type filtration:

$$X_0 \subset X_1 \subset \cdots \subset X_{n-2} \subset X_{n-1} = X$$

$X_i$ is the union of all $\leqslant i$-dimensional orbits.

The quotient filtration:

$$Q_0 \subset Q_1 \subset \cdots \subset Q_{n-2} \subset Q_{n-1} = Q,$$

$Q_i = X_i/T$.

# Orbit spaces and sponges

## Theorem (A.18)

If $T^{n-1} \circlearrowright X^{2n}$ is in general position (+some weak assumptions), then

1. $Q_{n-1} = Q = X/T$ is a closed topological manifold of dimension $n+1$;
2. $\dim Q_i = i$ for all $i \leqslant n-2$;
3. $Z := Q_{n-2}$ is locally modelled by $(n-2)$-skeleton $C_{n-2}$ of the fan of $\mathbb{C}P^{n-1}$.

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## Definition

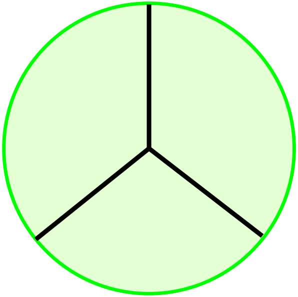
The closure of a connected component of $Q_i \backslash Q_{i-1}$ is called a face of $Q$.

## Definition

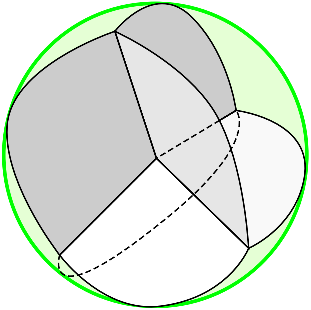
A space $Z$ is called $(n-2)$-dimensional sponge, if it is locally modelled by $C_{n-2}$. Abstract sponges also have faces, defined topologically.

# Orbit spaces and sponges

n=3



n=4



Local structure of an $(n-2)$-dimensional sponge.

# Equivariantly formal actions

## Definition

An $(n-2)$-dimensional sponge $Z$ is called acyclic if (1) $\widetilde{H}_*(F) = 0$ for any face $F$ of $Z$; (2) $\widetilde{H}_i(Z) = 0$ for $i \leqslant n-3$ (i.e. $Z$ is a Cohen–Macaulay space).

## Theorem (A.–Masuda'19)

If $T^{n-1} \circlearrowright X^{2n}$ is an equivariantly formal action in general position, then

- $Q$ is a homology $(n+1)$-sphere: $\widetilde{H}_i(Q) = 0$ for $i \leqslant n$;
- The sponge $Z = Q_{n-2}$ is acyclic.

If, moreover, all stabilizers are connected, these two conditions imply equivariant formality (over $\mathbb{Z}$).

# Combinatorics of sponges

## Definition

Let $Z$ be an acyclic sponge. Let $f_i$ denote the number of its $i$-dimensional faces, and $b = \operatorname{rk} \widetilde{H}_{n-2}(Z)$, the only nonzero Betti number of $Z$. The tuple $((f_0, \ldots, f_{n-2}), b)$ is called the extended $f$-vector of $Z$.

## Remark

$f_0 - f_1 + \cdots + (-1)^{n-2} f_{n-2} = 1 + (-1)^{n-2} b$ since both are equal to $\chi(Z)$. So far, $b$ can be expressed from $f_i$'s.

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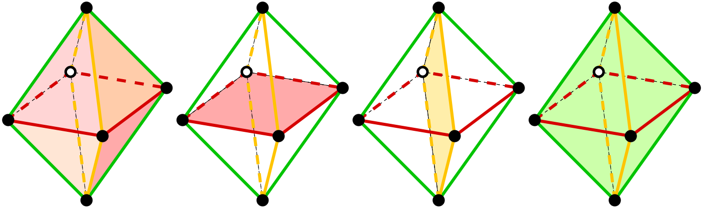
## Theorem (A.–Masuda'19)

If $((f_0, \ldots, f_{n-2}), b)$ is the extended f-vector of the sponge of an equivariantly formal action $T^{n-1} \circlearrowright X^{2n}$ in general position, then

$$\sum_{i=0}^{n} \beta_{2i}(X) t^{2i} = \sum_{i=0}^{n-2} f_i t^{2n-2i} (1 - t^2)^i + (1 + bt^2)(1 - t^2)^{n-1}.$$

# Action of $T^3$ on the Grassmann manifold $G_{4,2}$.

Buchstaber–Terzic'14: $G_{4,2}/T^3 \cong S^5$. The sponge:

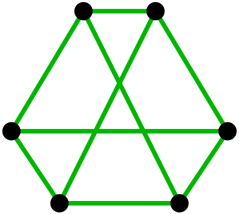


Extended $f$-vector $= ((6, 12, 11), 4)$. We have

$$\mathrm{Hilb}(H^*(G_{4,2}); t) = 6t^8 + 12t^6(1-t^2) + 11t^4(1-t^2)^2 + (1+4t^2)(1-t^2)^3 = 1 + t^2 + 2t^4 + t^6 + t^8.$$

# Action of $T^2$ on the full flag manifold $F_3$.

Buchstaber–Terzic'14-18: $F_3/T^2 \cong S^4$. The sponge:

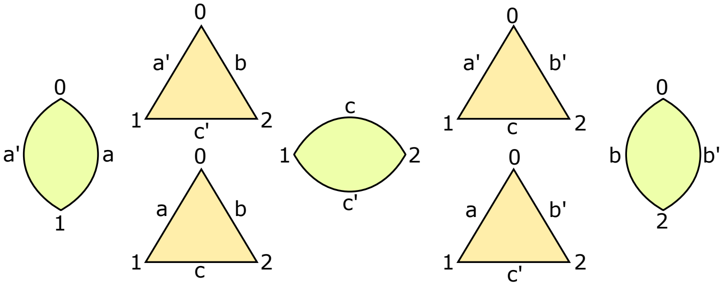


Extended $f$-vector $= ((6,9),4)$. We have

$$\mathrm{Hilb}(H^*(F_3);t) = 6t^6 + 9t^4(1-t^2) + (1+4t^2)(1-t^2)^2 = 1 + 2t^2 + 2t^4 + t^6.$$

# Action of $T^3$ on the quaternionic projective plane $\mathbb{H}P^2$.

Ayzenberg'19: $\mathbb{H}P^2/T^3 \cong S^5$. The sponge of $\mathbb{H}P^2$ is the quotient of the sponge of $G_{4,2}$ by the antipodal involution.




Extended $f$-vector $= ((3,6,7),3)$. We have

$$\mathrm{Hilb}(H^*(\mathbb{H}P^2);t) = 3t^8 + 6t^6(1-t^2) + 7t^4(1-t^2)^2 + (1+3t^2)(1-t^2)^3 = 1 + t^4 + t^8.$$

# Problems and questions

## Definition

Let us define the $h$-vector of an acyclic $(n-2)$-sponge by

$$\sum_{i=0}^{n} h_i t^{2i} = \sum_{i=0}^{n-2} f_i t^{2n-2i}(1-t^2)^i + (1+bt^2)(1-t^2)^{n-1}.$$

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## Problem 1

Prove "Dehn–Sommerville relations": $h_i = h_{n-i}$ for all acyclic sponges.

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## Problem 1

Prove "Dehn–Sommerville relations": $h_i = h_{n-i}$ for all acyclic sponges.

## Problem 2

Prove "Lower bound theorem": $h_i \geqslant 0$ for all acyclic sponges.

# Problems and questions

## Problem 3 (implies Problems 1 and 2)

Invent "the face algebra" of an acyclic sponge. That is, for an acyclic sponge $Z$, define a graded algebra $\Bbbk[Z]$ with the properties:

- In general, $\operatorname{Hilb}(\Bbbk[Z];t) = \dfrac{h_0 + h_1t^2 + \cdots + h_nt^{2n}}{(1-t^2)^n}$.
- In general, $\Bbbk[Z]$ is Gorenstein.
- $H_T^*(X;\Bbbk) \cong \Bbbk[Z]$, where $Z$ is the sponge of equivariantly formal complexity 1 action in general position on $X$.
- $H^*(X;\Bbbk) \cong \Bbbk[Z]/(l.s.o.p.)$.

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## Problem 4

Analogues of everything for real torus actions of complexity one: $\mathbb{Z}_2^{n-1} \circlearrowright X^n$. What should be "equivariant formality" in this case?

Thank you for listening!

# References


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